

Differentiability of quadratic BSDEs generated by continuous martingales

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Abstract

In this paper we consider a class of BSDEs with drivers of quadratic growth, on a stochastic basis generated by continuous local martingales. We first derive the Markov property of a forward-backward system (FBSDE) if the generating martingale is a strong Markov process. Then we establish the differentiability of a FBSDE with respect to the initial value of its forward component. This enables us to obtain the main result of this article, namely a representation formula for the control component of its solution. The latter is relevant in the context of securitization of random liabilities arising from exogenous risk, which are optimally hedged by investment in a given financial market with respect to exponential preferences. In a purely stochastic formulation, the control process of the backward component of the FBSDE steers the system into the random liability, and describes its optimal derivative hedge by investment in the capital market the dynamics of which is given by the forward component. The representation formula of the main result describes this delta hedge in terms of the derivative of the BSDE's solution process on the one hand, and the correlation structure of the internal uncertainty captured by the forward process and the external uncertainty responsible for the market incompleteness on the other hand. The formula extends the scope of validity of the results obtained by several authors in the Brownian setting. It is designed to extend a genuinely stochastic representation of the optimal replication in cross hedging insurance derivatives from the classical Black-Scholes model to incomplete markets on general stochastic bases. In this setting, Malliavin's calculus which is required in the Brownian framework is replaced by new tools based on techniques related to a calculus of quadratic covariations of basis martingales.

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1 Introduction

In recent years Backward Stochastic Differential Equations (BSDEs for short) with drivers of quadratic growth have shown to be relevant in several fields of application, e.g. the study of properties of PDEs (see *e.g.* [19, 5]). Closer to the subject of this work, they were employed to provide a genuinely stochastic approach to describe optimal investment strategies in a financial market in problems of hedging derivatives or liabilities of a small trader whose business depends on market external risk. The latter scenario was addressed for instance in [12, 2, 3, 18]. A small trader, such as an energy retailer, has a natural source of income deriving from his usual business. For instance, he may have a random position of revenues from heating oil sales at the end of a heating season. To (cross) hedge his risk arising from the partly market external uncertainty present in the temperature process during the heating season, for example via derivatives written on temperature, he decides to invest in the capital market the inherent uncertainty of which is only correlated with this index process. If the agent values his total income at terminal time by exponential utility, or his risk by the entropic risk measure, he may be interested in finding an optimal investment strategy that maximizes his terminal utility resp. minimizes his total risk. The description of such strategies, even under convex constraints for the set of admissible ones, is classical and may be achieved by convex duality methods, and formulated in terms of the analytic Hamilton-Jacobi-Bellman equation. In a genuinely stochastic approach, [12] interpreted the martingale optimality principle by means of BSDEs with drivers of quadratic growth to come up with a solution of this optimal investment problem even under closed constraints that are not necessarily convex. The optimal investment strategy is described by the control process in the solution pair of such a BSDE with an explicitly known driver. Using this approach the authors of [3] investigate utility indifference prices and delta hedges for derivatives or liabilities written on non-tradable underlyings such as temperature in incomplete financial market models. A sensitivity analysis of the dependence of the optimal investment strategies on the initial state of the Markovian forward process modeling the external risk process provides an explicit delta hedging formula from the representation of indifference prices in terms of forward-backward systems of stochastic differential equations (FBSDEs). In the framework of a Brownian basis, this analysis requires both the parametric as well as variational differentiability in the sense of Malliavin calculus of the solutions of the BSDE part (see [2, 3, 5]). Related optimal investment problems have been investigated in situations in which the Gaussian basis is replaced by the one of a continuous martingale ([17] and [18], see also [13]).

In this paper we intend to extend this utility indifference based explicit description of a delta hedge to much more general stochastic bases. Our main result will provide a probabilistic representation of the optimal delta hedge of [3], obtained there in the Brownian setting, to more general scenarios in which pricing rules are based on general continuous local martingales. We do this through a sensitivity analysis of related systems of FBSDEs

on a stochastic basis created by a continuous local martingale. As the backward component of our system, we consider a BSDE of the form (1.1) driven by a continuous local martingale M with dynamics

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T f(s, Y_s, Z_s) dC_s - \int_t^T dL_s + \frac{\kappa}{2} \int_t^T d\langle L, L \rangle_s, \quad t \in [0, T], \quad (1.1)$$

where the *generator* f is assumed to be quadratic as a function of Z , the *terminal condition* B is bounded, C is an increasing process defined as $C := \arctan(\sum_i \langle M^{(i)}, M^{(i)} \rangle)$, L is a martingale orthogonal to M with quadratic variation $\langle L, L \rangle$ and κ is a positive constant. A solution of (1.1) is given by a triplet (Y, Z, L) . The forward component of our system is of the form

$$X_s = x + \int_0^s \sigma(r, X_r, M_r) dM_r + \int_0^s b(r, X_r, M_r) dC_r, \quad s \in [0, T]. \quad (1.2)$$

We first prove in Theorem 3.4 that the solution processes Y and Z satisfy the Markov property, provided the terminal condition B is a smooth function of the terminal value of the forward process (1.2) and that the local martingale M is a strong Markov process. There is a subtlety in this setting which goes beyond causing a purely technical complication, namely that only the pair (X, M) is a Markov process (as proved for example in [21, 7, 22]). Only if M has independent increments it is a stand-alone Markov process. We then show in Theorem 4.6 that the process Y is differentiable with respect to the initial value of the forward component (1.2) and that the derivatives of Y and Z again satisfy a BSDE. The two properties then combine to allow us to state and prove the main contribution of this paper. Thereby our delta hedge representation (Theorem 5.1) generalizes the formula obtained in the Gaussian setting, see [3, Theorem 6.7] for the quadratic case and [14, Corollary 4.1] for the Lipschitz case. More precisely, we show that there exists a deterministic function u such that

$$Z_s = \partial_2 u(s, X_s, M_s) \sigma(s, X_s, M_s) + \partial_3 u(s, X_s, M_s), \quad (1.3)$$

where $Y_s = u(s, X_s, M_s)$, $s \in [0, T]$, and ∂_i denotes the partial derivative with respect to the i -th variable (see Theorem 5.1). In addition we show that if M has independent increments and the coefficients of the forward process do not depend on M , then $Y_s = u(s, X_s)$ and equality (1.3) becomes $Z_s = \partial_2 u(s, X_s) \sigma(s, X_s)$ which coincides with the formula known for the case in which M is a Brownian motion. To the best of our knowledge, relation (1.3) is known only in the Brownian setting and the proof used in the literature relies on the representation of the stochastic process Z as the trace of the Malliavin derivative D (*i.e.* $Z_s = D_s Y_s$, $s \in [0, T]$) relative to the underlying Brownian motion. Since Malliavin's calculus is not available for general continuous local martingales, we propose a new approach based on stochastic calculus techniques, in which directional variational derivatives of Malliavin's calculus are replaced by absolute continuity properties of mixed variation processes of local basis martingales. Note also reference [4], where a Markovian representation of the solution (Y, Z) of the solution of a BSDE driven by a symmetric Markov process is given and whose driver is Lipschitz in z and satisfies a monotonicity condition in y (see [4, Condition (H2), p. 35]). However the representation of the component Z is not exactly similar to our representation (compare [4, Theorem 5.4] and Theorem 5.1) due to a lack of regularity of

the BSDE's driver in the setting of [4]. Note finally that the method employed in [4] relies on the calculus of Fukushima (see [11]) for symmetric Markov processes. We finally emphasize that the local martingale M considered in this paper is not assumed to satisfy the martingale representation property.

The layout of this article is as follows. In Section 2 we state the main notations and assumptions used in the paper. We discuss the Markov property of an FBSDEs in Section 3. In Section 4 we give sufficient conditions on the FBSDEs to be differentiable in the initial values of its forward component, while Section 5 is devoted to the representation formula (1.3). Section 6 is devoted to the finance and insurance application of our main result.

2 Preliminaries

Notations

Let $(M_t)_{t \in [0, T]}$ be a continuous d -dimensional local martingale with $M_0 = 0$ which is defined on a probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ where T is a fixed positive real number. We assume that the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is continuous and complete so that every \mathbb{P} -martingale is of the form $Z \cdot M + L$, where Z is a predictable d -dimensional process and L a \mathbb{R} -valued martingale strongly orthogonal to M , i.e. $\langle L, M^{(i)} \rangle = 0$ for $i = 1, \dots, d$. Here and in the following $M^{(i)}$, $i = 1, \dots, d$, denotes the entries of the vector M . We assume that there exists a positive constant Q such that

$$\langle M^{(i)}, M^{(j)} \rangle_T \leq Q, \quad \forall 1 \leq i, j \leq d, \quad \mathbb{P} - a.s.$$

The Euclidean norm is denoted by $|\cdot|$ and with \mathcal{E} we refer to the stochastic exponential. From the Kunita-Watanabe inequality it follows that there exists a continuous, adapted, bounded and increasing real-valued process $(C_t)_{t \in [0, T]}$ and a $\mathbb{R}^{d \times d}$ -valued predictable process $(q_t)_{t \in [0, T]}$ such that the quadratic variation process $\langle M, M \rangle$ can be written as

$$\langle M, M \rangle_t = \int_0^t q_r q_r^* dC_r, \quad t \in [0, T],$$

where $*$ denotes the transposition. We choose as in [18], $C := \arctan \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle \right)$. We write \mathcal{P} for the predictable σ -field on $\Omega \times [0, T]$. Next we specify several spaces which we use in the sequel. Given the arbitrary non-negative and progressively measurable real-valued process $(\psi_t)_{t \in [0, T]}$ we define Ψ by $\Psi_t := \int_0^t \psi_s^2 dC_s, 0 \leq t \leq T$. For any $\beta > 0$, $n \in \mathbb{N}$ and $p \in [1, \infty)$ we set

- $\mathcal{S}^\infty := \{X : \Omega \times [0, T] \rightarrow \mathbb{R} \mid X \text{ bounded and continuous process}\},$
- $\mathcal{S}^p := \left\{ X : \Omega \times [0, T] \rightarrow \mathbb{R} \mid X \text{ predictable process and } \mathbb{E}[\sup_{t \in [0, T]} |X_t|^p] < \infty \right\},$
- $L^p(d\langle M, M \rangle \otimes d\mathbb{P})$
 $:= \left\{ Z : \Omega \times [0, T] \rightarrow \mathbb{R}^{1 \times d} \mid Z \text{ predictable process and } \mathbb{E}[(\int_0^T |q_s Z_s^*|^2 dC_s)^{\frac{p}{2}}] < \infty \right\},$
- $\mathcal{M}^2 := \{X : \Omega \times [0, T] \rightarrow \mathbb{R} \mid X \text{ square-integrable martingale}\},$

- $\mathbb{L}^\infty := \{\xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ } \mathcal{F}_T\text{-measurable bounded random variable}\},$
- $\mathbb{L}^p := \{\xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ } \mathcal{F}_T\text{-measurable random variable and } \mathbb{E}[|\xi|^p] < \infty\},$
- $\mathbb{L}_\beta^2(\mathbb{R}^{n \times 1}) := \{\xi : \Omega \rightarrow \mathbb{R} \mid \xi \text{ } \mathcal{F}_T\text{-measurable random variable and } \mathbb{E}[e^{\beta\Psi_T}|\xi|^2] < \infty\},$
- $\mathbb{H}_\beta^2 :=$
 $\left\{X : \Omega \times [0, T] \rightarrow \mathbb{R}^{d \times 1} \mid X \text{ predictable process and } \|X\|_\beta^2 := \mathbb{E}[\int_0^T e^{\beta\Psi_t}|X_t|^2 dC_t] < \infty\right\},$
- $\mathbb{S}_\beta^2 :=$
 $\left\{X : \Omega \times [0, T] \rightarrow \mathbb{R}^{d \times 1} \mid X \text{ continuous process and } \|X\|_\beta^2 := \mathbb{E}[\sup_{t \in [0, T]} e^{\beta\Psi_t}|X_t|^2] < \infty\right\}.$

Throughout this paper we will make use of the notation $(M^{t,m})_{s \in [t, T]}$ ($t < T$, $m \in \mathbb{R}^{d \times 1}$) which refers to the martingale

$$M_s^{t,m} := m + M_s - M_t$$

defined with respect to the filtration $(\mathcal{F}_s^t)_{s \in [t, T]}$ with $\mathcal{F}_s^t := \sigma(\{M_u - M_t, t \leq u \leq s\})$. Obviously, all the preceding definitions can be introduced with $M^{t,m}$ in place of M and will inherit the superscript t,m . For convenience, we write $M^m := M^{0,m}$.

FBSDEs driven by continuous martingales

In this subsection we present the main hypotheses needed in this paper. Let us fix $x \in \mathbb{R}^{n \times 1}$ and $m \in \mathbb{R}^{d \times 1}$ and consider the process $X^{x,m} := (X_t^{x,m})_{t \in [0, T]}$ which is defined as a solution of the following stochastic differential equation (SDE)

$$X_t^{x,m} = x + \int_0^t \sigma(s, X_s^{x,m}, M_s^m) dM_s + \int_0^t b(s, X_s^{x,m}, M_s^m) dC_s, \quad t \in [0, T], \quad (2.1)$$

where the coefficients $\sigma : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{n \times d}$ and $b : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{n \times 1}$ are Borel-measurable functions. By [9, Theorem 1] and [21, Theorem 3.1], this SDE has a unique solution if the following hypothesis is satisfied.

(HSDE) The functions σ and b are continuous in (s, x, m) and there exists a $K > 0$ such that for all $x_1, x_2 \in \mathbb{R}^{n \times 1}$ and $m_1, m_2 \in \mathbb{R}^{d \times 1}$

$$|\sigma(s, x_1, m_1) - \sigma(s, x_2, m_2)| + |b(s, x_1, m_1) - b(s, x_2, m_2)| \leq K(|x_1 - x_2| + |m_1 - m_2|),$$

for all $s \in [0, T]$. Next we give some properties of BSDEs which depend on the forward process $X^{x,m}$. More precisely we consider BSDEs of the form

$$\begin{aligned} Y_t^{x,m} = & F(X_T^{x,m}, M_T^m) - \int_t^T Z_r^{x,m} dM_r + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m}, q_r^*) dC_r - \int_t^T dL_r^{x,m} \\ & + \frac{\kappa}{2} \int_t^T d\langle L^{x,m}, L^{x,m} \rangle_r, \quad t \in [0, T], \end{aligned} \quad (2.2)$$

where $F : \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$ and $f : \Omega \times [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ are $\mathcal{B}(\mathbb{R}^{n \times 1})$ -respectively $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{n \times 1}) \otimes \mathcal{B}(\mathbb{R}^{d \times 1}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1 \times d})$ -measurable functions. By $\mathcal{B}(\mathbb{R}^d)$ we denote the Borel σ -algebra. A solution of the BSDE with *terminal condition* $F(X_T^{x,m})$, a constant κ and *generator* f is defined to be a triple of processes $(Y^{x,m}, Z^{x,m}, L^{x,m}) \in \mathcal{S}^\infty \times L^2(d\langle M, M \rangle \otimes d\mathbb{P}) \times \mathcal{M}^2$ satisfying (2.2) and such that $\langle L^{x,m}, M^i \rangle = 0$, $i = 1, \dots, d$, and \mathbb{P} -a.s. $\int_0^T |f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m}, q_r^*)| dC_r < \infty$.

Let $\mathcal{V} := \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \times \mathbb{R} \times \mathbb{R}^{1 \times d}$ and assume that (HSDE) holds. Furthermore we define the measure $\nu(A) = E[\int_0^T \mathbf{1}_A(s) dC_s]$ for all $A \in \mathcal{B}([0, T]) \otimes \mathcal{F}$. Under the following conditions existence and uniqueness of a solution of the backward equation (2.2) was recently discussed in [18, Theorem 2.5]:

(H1) The function F is bounded.

(H2) The generator f is continuous in (y, z) and there exists a nonnegative predictable process η such that $\int_0^T \eta_s dC_s \leq a$, where a is a positive constant as well as positive numbers b and γ , such that ν -a.e.

$$|f(s, x, m, y, z)| \leq \eta_s + b\eta_s|y| + \frac{\gamma}{2}|z|^2, \quad \text{with } \gamma \geq |\kappa|, \gamma \geq b, \quad (x, m, y, z) \in \mathcal{V}.$$

An additional assumption is needed to obtain uniqueness (see [18, Theorem 2.6]):

(H3) For every $\beta \geq 1$ we have $\int_0^T |f(s, 0, 0, 0, 0)| dC_s \in L^\beta(\mathbb{P})$. In addition, there exist two constants μ and ν , a nonnegative predictable process θ satisfying $\int_0^T |q_s \theta_s|^2 dC_s \leq c_\theta$ ($c_\theta \in \mathbb{R}$), such that ν -a.e.

$$(y_1 - y_2)(f(s, x, m, y_1, z) - f(s, x, m, y_2, z)) \leq \mu|y_1 - y_2|^2, \quad (x, m, y_i, z) \in \mathcal{V}, \quad i = 1, 2, \quad \text{and}$$

$$|f(s, x, m, y, z_1) - f(s, x, m, y, z_2)| \leq \nu(|q_s \theta_s| + |z_1| + |z_2|)|z_1 - z_2| \quad (x, m, y, z_i) \in \mathcal{V}, \quad i = 1, 2.$$

In this paper we will deal with martingales of bounded mean oscillation, briefly called BMO-martingales. We recall that $Z \cdot M$ is a BMO-martingale if and only if

$$\|Z \cdot M\|_{BMO_2} = \sup_{\tau \leq T} \mathbb{E} \left[\int_\tau^T |q_s Z_s^*|^2 dC_s \middle| \mathcal{F}_\tau \right]^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all stopping times $\tau \leq T$. We refer the reader to [15] for a survey. Specifically we need the following hypothesis.

(H4) There exist a $\mathbb{R}^{1 \times d}$ -valued predictable process K and a constant $\alpha \in (0, 1)$ such that $K \cdot M$ is a BMO martingale satisfying ν -a.e.

$$(y_1 - y_2)(f(s, x, m, y_1, z) - f(s, x, m, y_2, z)) \leq |q_s K_s^*|^{2\alpha} |y_1 - y_2|^2$$

for all $(x, m, y_i, z) \in \mathcal{V}$, $i = 1, 2$, and

$$|f(s, x, m, y, z_1) - f(s, x, m, y, z_2)| \leq |q_s K_s^*| |z_1 - z_2|$$

for all $(x, m, y, z_i) \in \mathcal{V}$, $i = 1, 2$.

Throughout this paper we also consider a second type of BSDEs also associated to the forward process $X^{x,m}$ solving (2.1), i.e.

$$U_t^{x,m} = F(X_T^{x,m}, M_T^m) - \int_t^T V_s^{x,m} dM_s + \int_t^T f(s, X_s^{x,m}, M_s^m, U_s^{x,m}, V_s^{x,m}, q_s^*) dC_s + \int_t^T dN_s^{x,m}, \quad (2.3)$$

$t \in [0, T]$, where N is a square-integrable martingale. This type of BSDEs has been studied by El Karoui and Huang in [13]. Under the following assumptions on terminal condition $F(X_T^{x,m}, M_T^m)$ and generator f , there exists a unique solution $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m}) \in \mathbb{S}_\beta^2 \times \mathbb{H}_\beta^2 \times \mathcal{M}^2$ to the BSDE 2.3:

(L1) The function F satisfies $F(X_T^{x,m}, M_T^m) \in \mathbb{L}_\beta^2(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ for some β large enough.

(L2) The generator f satisfies ν -a.e.

$$|f(s, x, m, y_1, z_1) - f(s, x, m, y_2, z_2)| \leq r_s |y_1 - y_2| + \theta_s |z_1 - z_2|, \quad (x, m, y_i, z_i) \in \mathcal{V}, i = 1, 2,$$

where r and θ are two non-negative predictable processes. Let $\alpha_s^2 = r_s + \theta_s^2$. We assume ν -a.e. that $\alpha_s^2 > 0$ and $\frac{f(\cdot, 0, 0)}{\alpha} \in \mathbb{H}_\beta^2$ for some $\beta > 0$ large enough.

We conclude this section by presenting assumptions which will be useful in Section 4, where we find sufficient conditions for a FBSDEs to be differentiable in its initial values $(x, m) \in \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1}$. Given a function $g : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$ we denote the partial derivatives with respect to the i -th variable by $\partial_i g(s, x, m)$ and, if no confusion can arise, we write $\partial_2 g(s, x, m) := (\partial_{1+j} g(s, x, m))_{j=1, \dots, n}$ and $\partial_3 g(s, x, m) := (\partial_{1+n+j} g(s, x, m))_{j=1, \dots, d}$. To simplify the notation we introduce for each pair (σ, b) of forward coefficients the second order differential operator

$$\begin{aligned} \mathcal{L} = & \sum_{i=1}^n b_i(\cdot) \partial_i + \frac{1}{2} \sum_{i,j=1}^n [(\sigma q) \cdot (\sigma q)^*]_{ij}(\cdot) \partial_{ij} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^d [\sigma q q^*]_{ij}(\cdot) \partial_{i,n+j} \\ & + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^n [q(\sigma q)^*]_{ij}(\cdot) \partial_{n+i,j} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d [q q^*]_{ij}(\cdot) \partial_{n+i,n+j}. \end{aligned}$$

(D1) The coefficients σ and b have locally Lipschitz partial derivatives in x and m uniformly in time.

(D2) The function F is twice differentiable and such that $\sum_{i=1}^n \partial_i F \sigma_{ij} + \partial_{n+j} F$ for all $j = 1, \dots, d$, and $\mathcal{L}F$ have locally Lipschitz partial derivatives.

(D3) The generator f is differentiable in x, m, y and z and there exist a constant $C > 0$ and a nonnegative predictable process θ satisfying $\int_0^T |q_s \theta_s|^2 dC_s \leq c_\theta$ ($c_\theta \in \mathbb{R}$), such that the partial derivatives satisfy ν -a.e.

$$|\partial_i f(s, x, m, y, z)| \leq C(|q_s \theta_s| + |z|), \quad (x, m, y, z) \in \mathcal{V}, \quad i = 2, \dots, 5.$$

- (D4) The generator f is differentiable in x, m, y and z and there exist a constant $C > 0$ and a nonnegative predictable process θ satisfying $\int_0^T |q_s \theta_s|^2 dC_s \leq c_\theta$ ($c_\theta \in \mathbb{R}$), such that the partial derivative $\partial_5 f$ is Lipschitz in (x, m, y, z) and for all $i = 2, \dots, 4$ the following inequality holds ν -a.e.

$$\begin{aligned} & |\partial_i f(s, x_1, m_1, y_1, z_1) - \partial_i f(s, x_2, m_2, y_2, z_2)| \\ & \leq C(|q_s \theta_s| + |z_1| + |z_2|)(|x_1 - x_2| + |m_1 - m_2| + |y_1 - y_2| + |z_1 - z_2|), \end{aligned}$$

for all $(x_j, m_j, y_j, z_j) \in \mathcal{S}, j = 1, 2$.

3 The Markov property of FBSDEs

For a fixed initial time $t \in [0, T)$ and initial values $x \in \mathbb{R}^{n \times 1}$ and $m \in \mathbb{R}^{d \times 1}$ we consider a SDE of the form

$$X_s^{t,x,m} = x + \int_t^s \sigma(u, X_u^{t,x,m}, M_u^{t,m}) dM_u + \int_t^s b(u, X_u^{t,x,m}, M_u^{t,m}) dC_u, \quad s \in [t, T], \quad (3.1)$$

where M is a local martingale as in Section 2 with values in $\mathbb{R}^{d \times 1}$, $\sigma : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{n \times d}$ and $b : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{n \times 1}$. Throughout this chapter the coefficients σ and b satisfy (HSDE) and hence (3.1) has a unique solution $X^{t,x,m}$. Before stating and proving the main results of this section we recall the following proposition which is a combination of [7, Theorem (8.11)] (see also [22, V.Theorem 35]) and [20, Theorem 5.3].

Proposition 3.1. *i) If M is a strong Markov process then $(M_s^{t,m}, X_s^{t,x,m})_{s \in [t, T]}$ is a strong Markov process.*

ii) If M is a strong Markov process with independent increments and if the coefficients σ and b do not depend on M , that is to say

$$X_s^{t,x} = x + \int_t^s \sigma(u, X_u^{t,x}) dM_u + \int_t^s b(u, X_u^{t,x}) dC_u,$$

then the process $(X_s^{t,x})_{s \in [t, T]}$ itself is a strong Markov process.

Note that in [2, 3, 14] the martingale considered is a standard Brownian motion so that situation *ii)* of the proposition above applies. In fact this case presents at least two major advantages, firstly the process X is a Markov process itself and secondly the quadratic variation of M is deterministic.

This section is organized as follows. We first prove in Proposition 3.2 that the solution of a Lipschitz BSDEs associated to a forward SDE of the form (3.1) is already determined by the solution $X^{t,x,m}$ of (3.1) and the Markov process $M^{t,m}$. In Theorem 3.4 we then extend this result to quadratic BSDEs.

Consider a BSDEs of the form

$$U_s^{t,x,m} = F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T V_r^{t,x,m} dM_r + \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{t,x,m}, V_r^{t,x,m} q_r^*) dC_r$$

$$- \int_s^T dN_r^{t,x,m}, \quad s \in [t, T]. \quad (3.2)$$

We suppose that the driver does not depend on Ω and hence is a deterministic Borel measurable function $f : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$. If F and f satisfy hypotheses (L1) and (L2) then the BSDE (3.2) admits a unique solution $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m}) \in \mathbb{S}_\beta^2 \times \mathbb{H}_\beta^2 \times \mathcal{M}^2$ (see [13, Theorem 6.1]). By $\mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ we denote the σ -algebra generated by the family of functions $(x, m) \mapsto \mathbb{E} \left[\int_t^T \phi(s, X_s^{t,x,m}, M_s^{t,m}) dC_s \right]$, where $\phi : \Omega \times [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$ is predictable, continuous and bounded.

Proposition 3.2. *Assume that M is a strong Markov process and that (L1) and (L2) are in force. Then there exist deterministic functions $u : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ -measurable and $v : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{1 \times d}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ -measurable such that*

$$U_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad V_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}), \quad s \in [t, T]. \quad (3.3)$$

Remark 3.3. *Before turning to the proof of Proposition 3.2 we stress the following point. Assume M and X are as in Proposition 3.1 ii) and that the driver f in (3.2) does not depend on M , then Proposition 3.2 is equivalent to the existence of deterministic functions $u : [0, T] \times \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1})$ -measurable and $v : [0, T] \times \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}^{1 \times d}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1})$ -measurable such that*

$$U_s^{t,x} = u(s, X_s^{t,x}), \quad V_s^{t,x} = v(s, X_s^{t,x}), \quad s \in [t, T].$$

Proof of Proposition 3.2. Consider the following sequence $(U^{k,t,x,m}, V^{k,t,x,m}, N^{k,t,x,m})_{k \geq 0}$ of BSDEs

$$\begin{aligned} U^{0,t,x} &= V^{0,t,x} = 0, \\ U_s^{k+1,t,x} &= F(X_T^{t,x,m}, M_T^{t,m}) + \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{k,t,x,m}, V_r^{k,t,x,m}, q_r^*) dC_r \\ &\quad - \int_s^T V_r^{k+1,t,x,m} dM_r - \int_s^T dN_r^{k+1,t,x,m}. \end{aligned} \quad (3.4)$$

We recall an estimate obtained in [13, p. 35]. Let $\alpha > 0$ and $\beta > 0$ be as in Section 2. Then

$$\begin{aligned} &\|\alpha(U^{k+1,t,x,m} - U^{k,t,x,m})\|_\beta^2 + \|q(V^{k+1,t,x,m} - V^{k,t,x,m})^*\|_\beta^2 + \|(N^{k+1,t,x,m} - N^{k,t,x,m})\|_\beta^2 \\ &\leq \varepsilon \left(\|\alpha(U^{k,t,x,m} - U^{k-1,t,x,m})\|_\beta^2 + \|q(V^{k,t,x,m} - V^{k-1,t,x,m})^*\|_\beta^2 + \|(N^{k,t,x,m} - N^{k-1,t,x,m})\|_\beta^2 \right), \end{aligned}$$

where ε is a constant depending on β which can be chosen with $\varepsilon < 1$. Applying the result recursively we obtain

$$\begin{aligned} &\|\alpha(U^{k+1,t,x,m} - U^{k,t,x,m})\|_\beta^2 + \|q(V^{k+1,t,x,m} - V^{k,t,x,m})^*\|_\beta^2 + \|(N^{k+1,t,x,m} - N^{k,t,x,m})\|_\beta^2 \\ &\leq \varepsilon^k \left(\|\alpha(U^{1,t,x,m} - U^{0,t,x,m})\|_\beta^2 + \|q(V^{1,t,x,m} - V^{0,t,x,m})^*\|_\beta^2 + \|(N^{1,t,x,m} - N^{0,t,x,m})\|_\beta^2 \right). \end{aligned}$$

Since

$$\sum_{k=0}^{\infty} \|\alpha(U^{k+1,t,x,m} - U^{k,t,x,m})\|_{\beta}^2 + \|q(V^{k+1,t,x,m} - V^{k,t,x,m})^*\|_{\beta}^2 + \|(N^{k+1,t,x,m} - N^{k,t,x,m})\|_{\beta}^2 < \infty$$

the sequence $(U^{k,t,x,m}, V^{k,t,x,m}, N^{k,t,x,m})_k$ converges ν -a.e. to $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$ as k tends to infinity.

We show by induction on $k \geq 1$ the following property $(Prop_k)$:

$(Prop_k)$ There exist deterministic functions $\Phi^k : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ -measurable and $\Psi^k : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{1 \times d}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ -measurable such that $U_s^{k,t,x,m} = \Phi^k(s, X_s^{t,x,m}, M_s^{t,m})$ and $V_s^{k,t,x,m} = \Psi^k(s, X_s^{t,x,m}, M_s^{t,m})$, for $t \leq s \leq T, k \in \mathbb{N}$.

Proof of $(Prop_1)$: From the definition of $U^{1,t,x,m}$ and since $N^{1,t,x,m}$ is a martingale we have for $s \in [t, T]$

$$\begin{aligned} U_s^{1,t,x,m} &= \mathbb{E} [U_s^{1,t,x,m} | \mathcal{F}_s^t] \\ &= \mathbb{E} \left[F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0) dC_r | \mathcal{F}_s^t \right]. \end{aligned} \quad (3.5)$$

The Markov property and Doob-Dynkin's Lemma give

$$\begin{aligned} U_s^{1,t,x,m} &= \mathbb{E} \left[F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0) dC_r | \mathcal{F}_s^t \right] \\ &= \mathbb{E} \left[F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0) dC_r | (X_s^{t,x,m}, M_s^{t,m}) \right] \\ &= \Phi^1(s, X_s^{t,x,m}, M_s^{t,m}), \end{aligned}$$

where $\Phi^1 : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$. Now let

$$R_s^{1,t,x,m} = U_s^{1,t,x,m} + \int_t^s f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0) dC_r, \quad s \in [t, T].$$

Then for $s \in [t, T]$

$$R_s^{1,t,x,m} = \int_t^s V_r^{1,t,x,m} dM_r + N_s^{t,x,m} - N_t^{t,x,m}, \quad (3.6)$$

hence using the localization technique we can assume that $R^{1,t,x,m}$ is a strongly additive (in the sense of [7, p. 169]) square integrable martingale. Now we apply [6, Theorem (2.16)] to $\mathcal{Y}^1 := M$ and $\mathcal{Y}^2 := R$. Thus there exist two additive locally square integrable martingales \mathcal{M}^1 and \mathcal{M}^2 , two deterministic functions $\Psi^1, \Psi^2 : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{1 \times d}$ such that $\mathcal{Y}^1 = \mathcal{M}^1$ and $\mathcal{Y}^2 = \int_t^s \Psi^1(s, X_s^{t,x,m}, M_s^{t,m}) d\mathcal{M}_s^1 + \int_t^s \Psi^2(s, X_s^{t,x,m}, M_s^{t,m}) d\mathcal{M}_s^2$. By definition of R we deduce that \mathcal{M}^2 has to be equal to $N^{1,t,x,m}$ (showing that $N^{1,t,x,m}$ is additive) and that $\Psi^2 \equiv 1$. This shows that

$$V_s^{1,t,x,m} = \Psi^1(s, X_s^{t,x,m}, M_s^{t,m}), \quad \nu - a.e..$$

Let $k \geq 1$, we prove $(Prop_k) \implies (Prop_{k+1})$. For $s \in [t, T]$ we have

$$\begin{aligned}
U_s^{k+1,t,x,m} &= \mathbb{E} \left[U_s^{k+1,t,x,m} | \mathcal{F}_s^t \right] \\
&= \mathbb{E} \left[F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{k,t,x,m}, V_r^{k,t,x,m} q_r^*) dC_r | \mathcal{F}_s^t \right] \\
&= \mathbb{E} \left[F(X_T^{t,x,m}, M_T^{t,m}) \right. \\
&\quad \left. - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, \Phi^k(r, X_r^{t,x,m}, M_r^{t,m}), \Psi^k(r, X_r^{t,x,m}, M_r^{t,m}) q_r^*) dC_r | \mathcal{F}_s^t \right] \\
&= \mathbb{E} \left[F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T f^k(r, X_r^{t,x,m}, M_r^{t,m}) dC_r | \mathcal{F}_s^t \right],
\end{aligned}$$

where $f^k(r, y, z) := f(r, y, \Phi^k(r, y, z), \Psi^k(r, y, z) q_r^*)$. Using the same argument as in the case $k = 1$ we deduce that there exists a function $\Phi^{k+1} : [0, T] \times \mathbb{R}^{m \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$ such that

$$U_s^{k+1,t,x,m} = \Phi^{k+1}(s, X_s^{t,x,m}, M_s^{t,m}).$$

For $s \in [t, T]$ let

$$R_s^{k+1,t,x,m} = U_s^{k+1,t,x,m} + \int_t^s f^k(r, X_r^{t,x,m}, M_r^{t,m}) dC_r - N_s^{k+1,t,x,m} + N_t^{k+1,t,x,m}.$$

Following the same procedure as before, we deduce that there exists a function $\Psi^{k+1} : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{1 \times d}$ such that

$$V_s^{k+1,t,x,m} = \Psi^{k+1}(s, X_s^{t,x,m}, M_s^{t,m}).$$

Let

$$u(r, y, z) := \limsup_{k \rightarrow \infty} \Phi^k(r, y, z), \quad v(r, y, z) := \limsup_{k \rightarrow \infty} \Psi^k(r, y, z).$$

Since the sequence $(U^{k,t,x}, V^{k,t,x}, N^{t,x,m})_k$ converges ν -a.e. to $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$ as k tends to infinity we have for $s \in [t, T]$

$$\begin{aligned}
u(s, X_s^{t,x,m}, M_s^{t,m}) &= (\limsup_{k \rightarrow \infty} \Phi^k)(s, X_s^{t,x,m}, M_s^{t,m}) = \limsup_{k \rightarrow \infty} (\Phi^k(s, X_s^{t,x,m}, M_s^{t,m})) \\
&= \limsup_{k \rightarrow \infty} U_s^{k,t,x,m} = U_s^{t,x,m}.
\end{aligned}$$

Similarly we obtain

$$v(s, X_s^{t,x,m}, M_s^{t,m}) = V_s^{t,x,m}.$$

□

We conclude this section by extending Proposition 3.2 to a quadratic FBSDE. More precisely we consider the following BSDEs

$$Y_s^{t,x,m} = F(X_T^{t,x,m}, M_T^{t,m}) - \int_s^T Z_u^{t,x,m} dM_u + \int_s^T f(u, X_u^{t,x,m}, M_u^{t,m}, Y_u^{t,x,m}, Z_u^{t,x,m} q_u^*) dC_u$$

$$- \int_s^T dL_u^{t,x,m} + \frac{\kappa}{2} \int_s^T d\langle L^{t,x,m}, L^{t,x,m} \rangle_u, \quad s \in [t, T], \quad (3.7)$$

where the forward process $X^{t,x,m}$ is a solution of (3.1). Again we suppose that the driver f does not depend on Ω and hence is a deterministic Borel measurable function $f : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$. If F satisfies (H1) and f hypotheses (H2) and (H3) then the BSDE (3.7) admits a unique solution $(Y^{t,x,m}, Z^{t,x,m}, L^{t,x,m}) \in \mathbb{S}^\infty \times L^2(d\langle M, M \rangle \otimes d\mathbb{P}) \times \mathcal{M}^2$ (see [18, Theorem 2.5]).

Theorem 3.4. *For $t \in [0, T]$ we assume that M is a strong Markov process and that (H1)-(H3) hold. Then there exist deterministic functions $u : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ -measurable and $v : [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}^{1 \times d}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1})$ -measurable such that*

$$Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad Z_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}), \quad s \in [t, T]. \quad (3.8)$$

Remark 3.5. *As mentioned in Remark 3.3, in the framework of Proposition 3.1 ii) when the driver f in (3.7) does not depend on M , Theorem 3.4 simplifies to the existence of deterministic functions $u : [0, T] \times \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1})$ -measurable and $v : [0, T] \times \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}^{1 \times d}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{n \times 1})$ -measurable such that*

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad Z_s^{t,x} = v(s, X_s^{t,x}), \quad s \in [t, T].$$

Proof of Theorem 3.4. Existence and uniqueness of the solution of (3.7) under the hypotheses (H1)-(H3) have been obtained in [18, Theorems 2.5 and 2.6]. More precisely it is shown in the proof of [18, Theorem 2.5] that the solution of a quadratic BSDE can be derived as the limit of solutions of a sequence of BSDEs with Lipschitz generators. We follow this proof and begin by relaxing condition (H2). Indeed consider the following assumption (H2').

(H2') The generator f is continuous in (y, z) and there exists a predictable process η such that $\eta \geq 0$ and $\int_0^T \eta_s dC_s \leq a$, where a is a positive constant. Furthermore there exists a constant $\gamma > 0$ such that ν -a.e.

$$|f(s, x, m, y, z)| \leq \eta_s + \frac{\gamma}{2} |z|^2, \quad \text{with } \gamma \geq |\kappa|, \quad (x, m, y, z) \in \mathcal{S}.$$

Assume that one can prove existence of a solution of (3.7) if f satisfies (H2') instead of (H2). Let f_K be the generator f truncated in Y at level K (as in [18, Lemma 3.1]). More precisely, set $f_K(s, x, m, y, z) := f(s, x, m, \rho(y)_K, z)$ with

$$\rho_K(y) := \begin{cases} -K, & \text{if } y < -K, \\ y, & \text{if } |y| \leq K, \\ K, & \text{if } y > K. \end{cases}$$

It is shown in [18, proof of Theorem 2.5, Step 1] that f_K satisfies (H2'). Hence by hypothesis there exists a triple of stochastic processes $(Y_K^{t,x,m}, Z_K^{t,x,m}, L_K^{t,x,m})$ which solves (3.7) with generator f_K . Due to a comparison argument and since f_K and f coincide along the sample paths of the solution $(Y_K^{t,x,m}, Z_K^{t,x,m}, L_K^{t,x,m})$, it also solves (3.7) with f satisfying (H2). As

a consequence our proof is finished if we show that (3.8) holds for the truncated generator f_K which satisfies (H2').

The next step is to consider a BSDE which is shown in [18] to be in one to one correspondence with the BSDE (3.7) and is obtained via an exponential coordinate change. We only give a brief survey and refer to [18, proof of Theorem 2.5, Step 2] for a complete treatment. Setting $U^{t,x,m} := e^{\kappa Y^{t,x,m}}$ transforms (3.7) into the following BSDE

$$\begin{aligned} U_s^{t,x,m} = & e^{\kappa F(X_T^{t,x,m})} - \int_s^T V_r^{t,x,m} dM_r + \int_s^T g(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{t,x,m}, V_r^{t,x,m}, q_r^*) dC_r \\ & - \int_s^T dN_r^{t,x,m}, \quad s \in [t, T]. \end{aligned} \quad (3.9)$$

We refer to a solution of this BSDE as $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$. Since f_K satisfies (H2'), the new generator

$$g(s, x, m, u, v) := \left(\kappa \rho_{c^2}(u) f_K \left(s, x, m, \frac{\ln(u \vee c^1)}{\kappa}, \frac{v}{\kappa(u \vee c^1)} \right) - \frac{1}{2(u \vee c^1)} |v|^2 \right),$$

$(x, m, u, v) \in \mathcal{S}$, satisfies (H2') (where c^1 and c^2 are two explicit constants given in [18, p. 135-136] depending only on $(a, \kappa, \|F\|_\infty, b)$ where we recall that a and b are the constants appearing in the Assumption (H2)) and the triple $(Y^{t,x,m}, Z^{t,x,m}, L^{t,x,m})$ with

$$Y^{t,x,m} := \frac{\log(U^{t,x,m})}{\kappa}, \quad Z^{t,x,m} := \frac{V^{t,x,m}}{\kappa U^{t,x,m}}, \quad L^{t,x,m} := \frac{1}{\kappa U^{t,x,m}} \cdot N^{t,x,m} \quad (3.10)$$

is well defined and is solution to (3.7) with generator f_K satisfying (H2').

To derive the existence of a solution of (3.9) an approximating sequence of BSDEs with Lipschitz generator g^p and terminal condition $e^{(\kappa F(X_T^{t,x,m}))}$ is introduced in such a way that g^p converges $d\nu$ -almost everywhere to g as p tends to infinity. We do not specify the explicit expression for g^p , since we only need that the sequence is increasing in y , implying the same property for the solution component $(U^{p,t,x,m})_{p \in \mathbb{N}}$. For more details we refer to [18, proof of Theorem 2.5, step 3].

Let $p \geq 1$. We consider the BSDE (3.9) with generator g^p and terminal condition $e^{(\kappa F(X_T^{t,x,m}))}$. Since g^p is Lipschitz continuous we know from [13, Theorem 6.1] that a unique solution $(U^{p,t,x,m}, V^{p,t,x,m}, N^{p,t,x,m})$ exists. Now we can apply Proposition 3.2 which provides deterministic functions a^p and b^p such that

$$U_s^{p,t,x,m} = a^p(s, X_s^{t,x,m}, M_s^{t,m}) \text{ and } V_s^{p,t,x,m} = b^p(s, X_s^{t,x,m}, M_s^{t,m}), \quad s \in [t, T].$$

A subsequence, for convenience again denoted by $(U^{p,t,x,m}, V^{p,t,x,m}, N^{p,t,x,m})_{p \in \mathbb{N}}$, converges almost surely (with respect to $d\nu$) to the solution $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$ of (3.9). Letting

$$a(s, y, m) := \liminf_{p \rightarrow \infty} a^p(s, y, m), \quad b(s, y, m) := \liminf_{p \rightarrow \infty} b^p(s, y, m),$$

$(s, y, m) \in [0, T] \times \mathbb{R}^{d \times 1} \times \mathbb{R}^{n \times 1}$, we conclude that $U_s^{t,x,m} = a(s, X_s^{t,x,m}, M_s^{t,m})$ and $V_s^{t,x,m} = b(s, X_s^{t,x,m}, M_s^{t,m})$, $s \in [t, T]$. Since $(U^{p,t,x,m})_{p \in \mathbb{N}}$ is increasing, we may set

$$u := \frac{\ln a}{\kappa}, \quad v := \frac{b}{\kappa a}.$$

Hence the result follows by (3.10) and the one to one correspondence. \square

4 Differentiability of FBSDEs

In this section we derive differentiability of the FBSDE of (2.1)-(2.2) with respect to the initial data x and m . The presence of the quantity $\langle L, L \rangle$ in the equation, where we recall that L is part of the solution of (2.2), prevents us from extending directly the usual techniques presented for example in [2, 3, 5]. Under an additional assumption (MRP) defined in Section 4.2 we deduce the differentiability of (2.2) from that of the auxiliary BSDE (4.1).

4.1 Differentiability of an auxiliary FBSDE

As mentioned above we first prove the differentiability of an auxiliary BSDE which will allow us to deduce the result for (2.2) in Section 4.2.

For every $(x, m) \in \mathbb{R}^{(n+d) \times 1}$ let us consider the following forward backward system of equations

$$\begin{aligned} X_t^{x,m} &= x + \int_0^t \sigma(r, X_r^{x,m}, M_r^m) dM_r + \int_0^t b(r, X_r^{x,m}, M_r^m) dC_r, \\ Y_t^{x,m} &= F(X_T^{x,m}, M_T^m) - \int_t^T Z_r^{x,m} dM_r + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*) dC_r, \end{aligned} \quad (4.1)$$

where M is a continuous local martingale in $\mathbb{R}^{d \times 1}$ satisfying the martingale representation property and C, q, σ, b, F, f are as described in Section 2. A solution of this system is given by the triple $(X^{x,m}, Y^{x,m}, Z^{x,m}) \in \mathcal{S}^p \times \mathcal{S}^\infty \times L^2(d\langle M, M \rangle \otimes d\mathbb{P})$ of stochastic processes. Note that the system (4.1) has a unique solution if the coefficients σ and b of the forward component satisfy (HSDE) and the terminal condition F and the generator f of the backward part satisfy (H1)-(H3).

In this section we will give sufficient conditions for the system (4.1) to be differentiable in $(x, m) \in \mathbb{R}^{(n+d) \times 1}$. Before turning to the backward SDE of the system we provide some material about the differentiability of the forward component obtained in [22, V.7].

Proposition 4.1. *Assume that σ and b satisfy (D1). Then for almost all $\omega \in \Omega$ there exists a solution $X^{x,m}(\omega)$ of (4.1) which is continuously differentiable in x and m . In addition the derivatives $D_{ik}^x := \frac{\partial}{\partial x_k} X^{(i)x,m}$, $i, k = 1, \dots, n$, and $D_{ik}^m := \frac{\partial}{\partial m_k} X^{(i)x,m}$, $i = 1, \dots, n$, $k = 1, \dots, d$, satisfy the following SDE for $t \in [0, T]$*

$$\begin{aligned} D_{ikt}^x &= \delta_{ik} + \sum_{\alpha=1}^d \sum_{j=1}^n \int_0^t \partial_{1+j} \sigma_{i\alpha}(s, X_s^{x,m}, M_s^m) D_{jks}^x dM_s^{(\alpha)} \\ &\quad + \sum_{j=1}^n \int_0^t \partial_{1+j} b^{(i)}(s, X_s^{x,m}, M_s^m) D_{jks}^x dC_s, \end{aligned} \quad (4.2)$$

$$\begin{aligned} D_{ikt}^m &= \sum_{\alpha=1}^d \sum_{j=1}^n \int_0^t \partial_{1+j} \sigma_{i\alpha}(s, X_s^{x,m}, M_s^m) D_{jks}^m dM_s^{(\alpha)} + \sum_{j=1}^n \int_0^t \partial_{1+j} b^{(i)}(s, X_s^{x,m}, M_s^m) D_{jks}^m dC_s \\ &\quad + \sum_{\alpha=1}^d \int_0^t \partial_{1+n+k} \sigma_{i\alpha}(s, X_s^{x,m}, M_s^m) dM_s^{(\alpha)} + \int_0^t \partial_{1+n+k} b^{(i)}(s, X_s^{x,m}, M_s^m) dC_s \end{aligned} \quad (4.3)$$

and $\frac{\partial}{\partial m_k} M^{(j)m} = \delta_{kj}$, $k, j = 1, \dots, d$. Furthermore for all $p > 1$ there exists a positive constant κ such that the following estimate holds

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{x, m} - X_t^{x', m'}|^{2p} \right] \leq \kappa (|x - x'|^2 + |m - m'|^2)^p. \quad (4.4)$$

Proof. Let $(\tilde{X}_t^{x, m})_{t \in [0, T]}$ be the stochastic process with values in $\mathbb{R}^{(1+n+d) \times 1}$ defined as

$$\tilde{X}_t^{x, m} = \begin{pmatrix} t \\ X_t^{x, m} \\ M_t^m \end{pmatrix}.$$

This process is the solution of the SDE

$$d\tilde{X}_t^{x, m} = \tilde{\sigma}(\tilde{X}_t^{x, m}) d\tilde{M}_t, \quad \tilde{X}_0^{x, m} = (0, x, m)$$

with

$$\tilde{\sigma}(\tilde{X}_t^{x, m}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma(\tilde{X}_t^{x, m}) & b(\tilde{X}_t^{x, m}) \\ 0 & I_d & 0 \end{pmatrix}, \quad \tilde{M}_t = \begin{pmatrix} t \\ M_t \\ C_t \end{pmatrix}.$$

According to [22, Theorem V.39] the derivatives D^x , D^m and $\frac{\partial}{\partial m_k} M^{(j)m}$, $k, j = 1, \dots, d$, exist and are continuous in x and m . In addition, formula [22, (D) p. 312] leads to (4.2) and (4.3). The estimate (4.4) follows immediately from [22, (**), p. 309]. \square

We now focus on the BSDE of system (4.1). Let $\tilde{x} := (x, m) \in \mathbb{R}^{(n+d) \times 1}$ and e_i , $i = 1, \dots, n + d$, the unit vectors in $\mathbb{R}^{(n+d) \times 1}$. For all \tilde{x} , $h \neq 0$ and $i \in \{1, \dots, n + d\}$ let $\xi^{\tilde{x}, h, i} = \frac{1}{h} (F(X_T^{\tilde{x} + h e_i}, M_T^{\tilde{x} + h e_i}) - F(X_T^{\tilde{x}}, M_T^{\tilde{x}}))$. Here it is implicit that $M^{\tilde{x}}$ only depends on the component m in $\tilde{x} = (x, m)$. The following Lemma will be needed later in order to prove the differentiability of the backward component. To simplify the notation we suppress the superscript i .

Lemma 4.2. *Suppose that (D1) and (D2) hold. Then for every $p > 1$ there exists a constant $\kappa > 0$, such that for all $\tilde{x}, \tilde{x}' \in \mathbb{R}^{(n+d) \times 1}$, $h, h' \neq 0$*

$$\mathbb{E} \left[|\xi^{\tilde{x}, h} - \xi^{\tilde{x}', h'}|^{2p} \right] \leq \kappa (|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^p \quad (4.5)$$

Proof. We know from Itô's formula

$$\begin{aligned} F(X_t^{\tilde{x}}, M_t^m) &= F(X_0^{\tilde{x}}, M_0^m) \\ &+ \sum_{j=1}^d \int_0^t \left(\sum_{i=1}^n \partial_i F(X_s^{\tilde{x}}, M_s^m) \sigma_{ij}(s, X_s^{\tilde{x}}, M_s^m) + \partial_{n+j} F(X_s^{\tilde{x}}, M_s^m) \right) dM_s^{(j)} \\ &+ \int_0^t \left(\sum_{i=1}^n \partial_i F(X_s^{\tilde{x}}, M_s^m) b_i(s, X_s^{\tilde{x}}, M_s^m) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij} F(X_s^{\tilde{x}}, M_s^m) [(\sigma q) \cdot (\sigma q)^*]_{ij}(s, X_s^{\tilde{x}}, M_s^m) \right) ds \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^d \partial_{i,n+j} F(X_s^{\tilde{x}}, M_s^m) [\sigma q q^*]_{ij}(s, X_s^{\tilde{x}}, M_s^m) \\
& + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^n \partial_{n+i,j} F(X_s^{\tilde{x}}, M_s^m) [q(\sigma q)^*]_{ij}(s, X_s^{\tilde{x}}, M_s^m) \\
& + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \partial_{n+i,n+j} F(X_s^{\tilde{x}}, M_s^m) [q q^*]_{ij}(s, X_s^{\tilde{x}}, M_s^m) \Big) dC_s.
\end{aligned}$$

Hence $F(X_t^{\tilde{x}}, M_t^m)$ is the solution of an SDE with coefficients

$$\begin{aligned}
\hat{\sigma}_j(s, \tilde{x}) &= \sum_{i=1}^n \partial_i F(\tilde{x}) \sigma_{ij}(s, \tilde{x}) + \partial_{n+j} F(\tilde{x}) \text{ for all } j = 1, \dots, d, \\
\hat{b}(s, \tilde{x}) &= \sum_{i=1}^n \partial_i F(\tilde{x}) b_i(s, \tilde{x}) + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij} F(\tilde{x}) [(\sigma q) \cdot (\sigma q)^*]_{ij}(s, \tilde{x}) \\
&+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^d \partial_{i,n+j} F(\tilde{x}) [\sigma q q^*]_{ij}(s, \tilde{x}) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^n \partial_{n+i,j} F(\tilde{x}) [q(\sigma q)^*]_{ij}(s, \tilde{x}) \\
&+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \partial_{n+i,n+j} F(\tilde{x}) [q q^*]_{ij}(s, \tilde{x}).
\end{aligned}$$

By (D2) the coefficients $\hat{\sigma}$ and \hat{b} have locally Lipschitz partial derivatives and thus together with (4.4) the estimate (4.5) follows. \square

The next Lemma shows that we can choose the family $(Y^{\tilde{x}})$ to be continuous in $\tilde{x} \in \mathbb{R}^{(n+d) \times 1}$.

Lemma 4.3. *Let (H1)-(H3) and (D1)-(D3) be satisfied. Then for all $p > 1$ there exists a constant $c > 0$, such that for all $\tilde{x}, \tilde{x}' \in \mathbb{R}^{(n+d) \times 1}$*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^{\tilde{x}} - Y_t^{\tilde{x}'}|^{2p} \right] + \mathbb{E} \left[\left(\int_0^T |q_t(Z_t^{\tilde{x}} - Z_t^{\tilde{x}'})^*|^2 dC_t \right)^p \right] \leq c |\tilde{x} - \tilde{x}'|^{2p}. \quad (4.6)$$

Furthermore for almost all $\omega \in \Omega$ there exists a solution $Y^{\tilde{x}}(\omega)$ of (4.1) which is continuous in $\tilde{x} \in \mathbb{R}^{(n+d) \times 1}$.

Proof. Let $\delta Y := Y^{\tilde{x}} - Y^{\tilde{x}'}$, $\delta Z := Z^{\tilde{x}} - Z^{\tilde{x}'}$, $\delta M := M^m - M^{m'}$ and $\delta X := X^{\tilde{x}} - X^{\tilde{x}'}$. We also set for $s \in [0, T]$

$$\begin{aligned}
A_r^Z &:= \int_0^1 \partial_5 f(r, X_r^{\tilde{x}}, M_r^m, Y_r^{\tilde{x}}, Z_r^{\tilde{x}'} q_r^* + \zeta(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'} q_r^*) d\zeta \\
A_r^Y &:= \int_0^1 \partial_4 f(r, X_r^{\tilde{x}}, M_r^m, Y_r^{\tilde{x}'} + \zeta(Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}), Z_r^{\tilde{x}'} q_r^*) d\zeta
\end{aligned}$$

$$\begin{aligned}
A_r^M &:= \int_0^1 \partial_3 f(r, X_r^{\tilde{x}}, M_r^{m'} + \zeta(M_r^m - M_r^{m'}), Y_r^{\tilde{x}'}, Z_r^{\tilde{x}'} q_r^*) d\zeta \\
A_r^X &:= \int_0^1 \partial_2 f(r, X_r^{\tilde{x}'} + \zeta(X_r^{\tilde{x}} - X_r^{\tilde{x}'}), M_r^{m'}, Y_r^{\tilde{x}'}, Z_r^{\tilde{x}'} q_r^*) d\zeta.
\end{aligned}$$

Considering the difference δY of the backward component in (4.1) we see that for $t \in [0, T]$

$$\begin{aligned}
\delta Y_t &= F(X_T^{\tilde{x}}, M_T^m) - F(X_T^{\tilde{x}'}, M_T^{m'}) - \int_t^T \delta Z_r dM_r \\
&\quad + \int_t^T [f(r, X_r^{\tilde{x}}, M_r^m, Y_r^{\tilde{x}}, Z_r^{\tilde{x}} q_r^*) - f(r, X_r^{\tilde{x}'}, M_r^{m'}, Y_r^{\tilde{x}'}, Z_r^{\tilde{x}'} q_r^*)] dC_r \\
&= F(X_T^{\tilde{x}}, M_T^m) - F(X_T^{\tilde{x}'}, M_T^{m'}) - \int_t^T \delta Z_r dM_r \\
&\quad + \underbrace{\int_t^T (\delta Z_r q_r^* A_r^Z + \delta Y_r A_r^Y + \delta M_r^* A_r^M + \delta X_r^* A_r^X) dC_r}_{=: g(r, \delta Y_r, \delta Z_r q_r^*)}
\end{aligned}$$

holds. Note that $(\delta Y, \delta Z)$ can be seen as a BSDE whose generator g satisfies (H4) and whose terminal condition $F(X_T^{\tilde{x}}, M_T^m) - F(X_T^{\tilde{x}'}, M_T^{m'})$ is bounded (see (H1)). More precisely we derive with (D3) and [18, Lemma 3.1] the existence of a constant c such that for all $y, y_1, y_2 \in \mathbb{R}$ and $z, z_1, z_2 \in \mathbb{R}^{1 \times d}$ ν -a.e.

$$\begin{aligned}
|g(r, y, z_1) - g(r, y, z_2)| &\leq |A_r^Z| |z_1 - z_2| \\
&\leq c(|q_r \theta_r| + |Z_r^{\tilde{x}'} q_r^*| + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'} q_r^*)|) |z_1 - z_2| \quad \text{and} \\
|g(r, y_1, z) - g(r, y_2, z)| &\leq |A_r^M| |y_1 - y_2| \\
&\leq c(|q_r \theta_r| + |Z_r^{\tilde{x}'} q_r^*|) |y_1 - y_2|.
\end{aligned}$$

Hence we can apply the a priori estimates of Lemma A.1 and hence we know that for every $p > 1$ there exist constants $q > 1$ and $c > 0$ such that

$$\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0, T]} |\delta Y_t|^{2p} \right] + \mathbb{E} \left[\left(\int_0^T |q_t \delta Z_t^*|^2 dC_t \right)^p \right] \\
&\leq c \mathbb{E} \left[|F(X_T^{\tilde{x}}, M_T^m) - F(X_T^{\tilde{x}'}, M_T^{m'})|^{2pq} + \left(\int_0^T |\delta M_r^* A_r^M + \delta X_r^* A_r^X| dC_r \right)^{2pq} \right]^{\frac{1}{q}}.
\end{aligned} \tag{4.7}$$

By condition (D3) and Hölder's inequality we get

$$\begin{aligned}
&\mathbb{E} \left[\left(\int_0^T |\delta M_r^* A_r^M + \delta X_r^* A_r^X| dC_r \right)^{2pq} \right] \\
&\leq c \mathbb{E} \left[\left(\int_0^T |\delta M_r|^2 dC_r \right)^{2pq} \right]^{\frac{1}{2}} \mathbb{E} \left[\left(\int_0^T (|q_r \theta_r| + |Z_r^{\tilde{x}'} q_r^*|)^2 dC_r \right)^{2pq} \right]^{\frac{1}{2}} \\
&\quad + c \mathbb{E} \left[\left(\int_0^T |\delta X_r|^2 dC_r \right)^{2pq} \right]^{\frac{1}{2}} \mathbb{E} \left[\left(\int_0^T (|q_r \theta_r| + |Z_r^{\tilde{x}'} q_r^*|)^2 dC_r \right)^{2pq} \right]^{\frac{1}{2}}
\end{aligned}$$

Note that $\mathbb{E} \left[\left(\int_0^T |q_r \theta_r|^2 dC_r \right)^{2pq} \right]$ is bounded by (D3). Furthermore $\mathbb{E} \left[\left(\int_0^T |Z_r^{\tilde{x}'} q_r^*|^2 dC_r \right)^{2pq} \right]$ is bounded, as is seen by applying Lemma A.1. Hence

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |\delta M_r^* A_r^M + \delta X_r^* A_r^X| dC_r \right)^{2pq} \right] &\leq c|m - m'|^{2pq} + C \mathbb{E} \left[\left(\sup_{t \in [0, T]} |\delta X_t|^2 C_T \right)^{2pq} \right]^{\frac{1}{2}} \\ &\leq c(|m - m'|^{2pq} + |\tilde{x} - \tilde{x}'|^{2pq}), \end{aligned}$$

where the last inequality is due to (4.4). Combining (4.7), condition (D2) and the last inequality we obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\delta Y_t|^{2p} \right] + \mathbb{E} \left[\left(\int_0^T |q_s \delta Z_s^*|^2 \right)^p \right] \leq c|\tilde{x} - \tilde{x}'|^{2p}.$$

Now Kolmogorov's Lemma (see Theorem 73, Chapter IV in [22]) implies that there exists a version of $(Y^{\tilde{x}})$ which is continuous in \tilde{x} for almost all $\omega \in \Omega$. \square

For all $h \neq 0, \tilde{x} \in \mathbb{R}^{(n+d) \times 1}, t \in [0, T]$ let $U_t^{\tilde{x}, h} = \frac{1}{h}(Y_t^{\tilde{x}+he_i} - Y_t^{\tilde{x}})$, $V_t^{\tilde{x}, h} = \frac{1}{h}(Z_t^{\tilde{x}+he_i} - Z_t^{\tilde{x}})$, $\Delta_t^{\tilde{x}, h} = \frac{1}{h}(X_t^{\tilde{x}+he_i} - X_t^{\tilde{x}})$, $\varpi_t^{\tilde{x}, h} = \frac{1}{h}(M_t^{\tilde{x}+he_i} - M_t^{\tilde{x}})$ (where it is implicit that $M^{\tilde{x}}$ depends only on the component m of $\tilde{x} = (x, m)$), and $\xi^{\tilde{x}, h} = \frac{1}{h}(F(X_T^{\tilde{x}+he_i}, M_T^{\tilde{x}+he_i}) - F(X_T^{\tilde{x}}, M_T^{\tilde{x}}))$. We define δU by $\delta U = U^{\tilde{x}, h} - U^{\tilde{x}', h'}$ and the processes δV , $\delta \Delta$, $\delta \varpi$, and $\delta \xi$ in an analogous way. We give estimates on the differences of difference quotients of the family $(Y^{\tilde{x}})$.

Lemma 4.4. *Let (H1)-(H3) and (D1)-(D4) be satisfied. Then for each $p > 1$ there exists a constant $c > 0$ such that for any $\tilde{x}, \tilde{x}' \in \mathbb{R}^{(n+d) \times 1}$ and $h, h' \neq 0$*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |U_t^{\tilde{x}, h} - U_t^{\tilde{x}', h'}|^{2p} \right] \leq c(|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^p. \quad (4.8)$$

Proof. This proof is similar to that of Lemma 4.3. By definition of $U^{\tilde{x}, h}$ and of $U^{\tilde{x}', h'}$ we have

$$\begin{aligned} U_t^{\tilde{x}, h} &= \xi^{\tilde{x}, h} - \int_t^T V_r^{\tilde{x}, h} dM_r \\ &\quad + \int_t^T \frac{1}{h} \left[f(r, X_r^{\tilde{x}+he_i}, M_r^{\tilde{x}+he_i}, Y_r^{\tilde{x}+he_i}, Z_r^{\tilde{x}+he_i} q_r^*) - f(r, X_r^{\tilde{x}}, M_r^{\tilde{x}}, Y_r^{\tilde{x}}, Z_r^{\tilde{x}} q_r^*) \right] dC_r. \end{aligned} \quad (4.9)$$

As in the proof of Lemma 4.3 we decompose the integrand in the last term of the right hand side of the equality above by writing

$$\begin{aligned} &\frac{1}{h}(f(r, X_r^{\tilde{x}+he_i}, M_r^{\tilde{x}+he_i}, Y_r^{\tilde{x}+he_i}, Z_r^{\tilde{x}+he_i} q_r^*) - f(r, X_r^{\tilde{x}}, M_r^{\tilde{x}}, Y_r^{\tilde{x}}, Z_r^{\tilde{x}} q_r^*)) \\ &= V_t^{\tilde{x}, h} q_r^* (A^Z)_r^{\tilde{x}, h} + U_t^{\tilde{x}, h} (A^Y)_r^{\tilde{x}, h} + \varpi_r^{\tilde{x}, h*} (A^M)_r^{\tilde{x}, h} + \Delta^{\tilde{x}, h} (A^X)_r^{\tilde{x}, h}, \end{aligned}$$

where A^Z, A^Y, A^M, A^X are defined as in the proof of Lemma 4.3, for instance

$$(A^Z)_r^{\tilde{x}, h} := \int_0^1 \partial_5 f(r, X_r^{\tilde{x}+he_i}, M_r^{\tilde{x}+he_i}, Y_r^{\tilde{x}+he_i}, Z_r^{\tilde{x}} q_r^* + \theta(Z_r^{\tilde{x}+he_i} - Z_r^{\tilde{x}}) q_r^*) d\theta.$$

Taking the difference of two equations of the form (4.9) we obtain that $(\delta U, \delta V)$ satisfies the BSDE

$$\begin{aligned} \delta U_t = & \delta \xi - \int_0^T \delta V_r dM_t + \int_t^T \delta V_r q_r^* (A^Z)_{\tilde{x},h}^{\tilde{x}',h'} + \delta U_r (A^Y)_{\tilde{x},h}^{\tilde{x}',h'} + \left[q_r^* ((A^Z)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Z)_{\tilde{x}',h'}^{\tilde{x},h}) V_r^{\tilde{x}',h'} \right. \\ & + U_r^{\tilde{x}',h'} ((A^Y)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Y)_{\tilde{x}',h'}^{\tilde{x},h}) + \varpi_r^{\tilde{x},h*} (A^M)_{\tilde{x},h}^{\tilde{x}',h'} - \varpi_r^{\tilde{x}',h'*} (A^M)_{\tilde{x}',h'}^{\tilde{x},h} \\ & \left. + \Delta^{\tilde{x},h*} (A^X)_{\tilde{x},h}^{\tilde{x}',h'} - \Delta^{\tilde{x}',h'*} (A^X)_{\tilde{x}',h'}^{\tilde{x},h} \right] dC_r. \end{aligned} \quad (4.10)$$

The generator of this BSDE satisfies condition (H4) due to assumption (D3) (details are similar to those of the proof of Lemma 4.3 and are left to the reader). By Lemma A.1 for every $p > 1$ there exist constants $q > 1$ and $c > 0$ such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |\delta U_t|^{2p} + \left(\int_0^T |q_s \delta V_s^*|^2 dC_s \right)^p \right] \\ & \leq c \mathbb{E} \left[|\delta \xi|^{2pq} + \left(\int_0^T |q_r^* ((A^Z)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Z)_{\tilde{x}',h'}^{\tilde{x},h})| |V_r^{\tilde{x}',h'}| + |U_r^{\tilde{x}',h'}| |((A^Y)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Y)_{\tilde{x}',h'}^{\tilde{x},h})| \right. \right. \\ & \quad \left. \left. + |\varpi_r^{\tilde{x},h*} (A^M)_{\tilde{x},h}^{\tilde{x}',h'} - \varpi_r^{\tilde{x}',h'*} (A^M)_{\tilde{x}',h'}^{\tilde{x},h}| + |\Delta^{\tilde{x},h*} (A^X)_{\tilde{x},h}^{\tilde{x}',h'} - \Delta^{\tilde{x}',h'*} (A^X)_{\tilde{x}',h'}^{\tilde{x},h}| dC_r \right)^{2pq} \right]^{1/q}. \end{aligned}$$

We estimate separately each part of the right hand side of the inequality presented. First, by Cauchy-Schwarz' inequality we have

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T |q_r^* ((A^Z)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Z)_{\tilde{x}',h'}^{\tilde{x},h})| |V_r^{\tilde{x}',h'}| dC_r \right)^{2pq} \right] \\ & \leq \mathbb{E} \left[\left(\int_0^T |q_r^* ((A^Z)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Z)_{\tilde{x}',h'}^{\tilde{x},h})|^2 dC_r \right)^{2pq} \right]^{1/2} \mathbb{E} \left[\left(\int_0^T |V_r^{\tilde{x}',h'}|^2 dC_r \right)^{2pq} \right]^{1/2} \\ & \leq C \mathbb{E} \left[\left(\int_0^T |q_r^* ((A^Z)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Z)_{\tilde{x}',h'}^{\tilde{x},h})|^2 dC_r \right)^{2pq} \right]^{1/2} \end{aligned}$$

since $\mathbb{E} \left[\left(\int_0^T |V_r^{\tilde{x}',h'}|^2 dC_r \right)^{2pq} \right]$ is bounded by Lemma A.1. Then hypothesis (D4) and a combination of Lemma 4.3 and (4.4) lead to the following estimate, in which the constant c may change from line to line

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T |q_r^* ((A^Z)_{\tilde{x},h}^{\tilde{x}',h'} - (A^Z)_{\tilde{x}',h'}^{\tilde{x},h})| |V_r^{\tilde{x}',h'}| dC_r \right)^{2pq} \right] \\ & \leq c \mathbb{E} \left[\left(\int_0^T |q_r^* (Z_r^{\tilde{x}} - Z_r^{\tilde{x}'})|^2 + |X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}|^2 + \underbrace{|M_r^{\tilde{x}+he_i} - M_r^{\tilde{x}'+h'e_i}|^2}_{=\tilde{x}+he_i - \tilde{x}' - h'e_i} \right. \right. \\ & \quad \left. \left. + |Y_r^{\tilde{x}+he_i} - Y_r^{\tilde{x}'+h'e_i}|^2 + |q_r^* (Z_r^{\tilde{x}+he_i} - Z_r^{\tilde{x}'+h'e_i})|^2 dC_r \right)^{2pq} \right]^{1/2} \end{aligned}$$

$$\leq c(|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^{pq}.$$

Similarly we derive

$$\mathbb{E} \left[\left(\int_0^T |U_r^{\tilde{x}', h'}| |(A^Y)_r^{\tilde{x}, h} - (A^Y)_r^{\tilde{x}', h'}| dC_r \right)^{2pq} \right] \leq c(|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^{pq}.$$

We next estimate, with constants c allowed to change from line to line

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T |\varpi_r^{\tilde{x}, h^*}(A^M)_r^{\tilde{x}, h} - \varpi_r^{\tilde{x}', h'^*}(A^M)_r^{\tilde{x}', h'}| dC_r \right)^{2pq} \right] \\ & \leq c \mathbb{E} \left[\left(\int_0^T \underbrace{|\varpi_r^{\tilde{x}, h} - \varpi_r^{\tilde{x}', h'}|}_{=0} |(A^M)_r^{\tilde{x}, h}| dC_r \right)^{2pq} \right] \\ & \quad + c \mathbb{E} \left[\left(\int_0^T \underbrace{|\varpi_r^{\tilde{x}', h'}|}_{=e_i} |(A^M)_r^{\tilde{x}, h} - (A^M)_r^{\tilde{x}', h'}| dC_r \right)^{2pq} \right] \\ & \leq c \mathbb{E} \left[\left(\int_0^T |(A^M)_r^{\tilde{x}, h} - (A^M)_r^{\tilde{x}', h'}| dC_r \right)^{2pq} \right] \\ & \leq c \mathbb{E} \left[\left(\int_0^T (|q_r \theta_r| + |Z_r^{\tilde{x}} q_s^*| + |Z_r^{\tilde{x}'} q_s^*|)^2 dC_r \right)^{2pq} \right]^{1/2} \mathbb{E} \left[\left(\int_0^T (|X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}| \right. \right. \\ & \quad \left. \left. + |Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}| + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'} q_s^*| + |M_r^{\tilde{x}} - M_r^{\tilde{x}'}| + |M_r^{\tilde{x}+he_i} - M_r^{\tilde{x}'+h'e_i}|)^2 dC_r \right)^{2pq} \right]^{1/2}, \end{aligned}$$

where the last inequality is due to hypotheses (D4) and Hölder's inequality. An application of the a priori estimates from Lemma A.1 implies that $\mathbb{E} \left[\left(\int_0^T (|q_r \theta_r| + |Z_r^{\tilde{x}} q_s^*| + |Z_r^{\tilde{x}'} q_s^*|)^2 dC_r \right)^{2pq} \right]$ is bounded. Then, using (4.4) and (4.6), we obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T |\varpi_r^{\tilde{x}, h^*}(A^M)_r^{\tilde{x}, h} - \varpi_r^{\tilde{x}', h'^*}(A^M)_r^{\tilde{x}', h'}| dC_r \right)^{2pq} \right] \\ & \leq c \mathbb{E} \left[\left(\int_0^T |X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}|^2 + |Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}|^2 + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'} q_s^*)^2 \right. \right. \\ & \quad \left. \left. + |M_r^{\tilde{x}} - M_r^{\tilde{x}'}|^2 + |M_r^{\tilde{x}+he_i} - M_r^{\tilde{x}'+h'e_i}|^2 dC_r \right)^{2pq} \right]^{1/2} \\ & \leq c(|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^{pq}. \end{aligned}$$

We now consider the last term whose treatment is similar to that of the term just discussed. We therefore give the main computations without providing detailed arguments. We have, with a constant C , the value of which may change from line to line,

$$\mathbb{E} \left[\left(\int_0^T |\Delta_r^{\tilde{x}, h^*}(A^X)_r^{\tilde{x}, h} - \Delta_r^{\tilde{x}', h'^*}(A^X)_r^{\tilde{x}', h'}| dC_r \right)^{2pq} \right]$$

$$\begin{aligned}
&\leq c\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'}|^2 dC_r\right)^{2pq}\right]^{1/2} \mathbb{E}\left[\left(\int_0^T |(A^X)_r^{\tilde{x},h}|^2 dC_r\right)^{2pq}\right]^{1/2} \\
&\quad + c\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x}',h'}| |(A^X)_r^{\tilde{x},h} - (A^X)_r^{\tilde{x}',h'}| dC_r\right)^{2pq}\right].
\end{aligned}$$

Using (D3) and Lemma A.1 we deduce that $\mathbb{E}\left[\left(\int_0^T |(A^X)_r^{\tilde{x},h}|^2 dC_r\right)^{2pq}\right]$ is bounded. Using hypothesis (D4) and (4.4) again we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h^*}(A^X)_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'^*}(A^X)_r^{\tilde{x}',h'}| dC_r\right)^{2pq}\right] \\
&\leq c\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'}|^2 dC_r\right)^{2pq}\right]^{1/2} \\
&\quad + c\mathbb{E}\left[\left(\int_0^T \left(|q_r\theta_r| + |Z_r^{\tilde{x}'}q_r^*| + |Z_r^{\tilde{x}}q_r^*|\right) \left(|X_r^{\tilde{x}} - X_r^{\tilde{x}'}| + |X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}| + |M_r^{\tilde{x}} - M_r^{\tilde{x}'}| \right.\right.\right. \\
&\quad \left.\left.\left.+ |Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}| + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'}q_r^*)| \right) dC_r\right)^{2pq}\right] \\
&\leq c(|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^{pq}.
\end{aligned}$$

We derive

$$\mathbb{E}[|\delta\xi|^{2pq}] \leq c(|\tilde{x} - \tilde{x}'|^2 + |h - h'|^2)^{pq}$$

from (4.5). This completes the proof of (4.8). \square

Proposition 4.5. *Let (H1)-(H3) and (D1)-(D4) be satisfied. Then there exists a solution $(X^{\tilde{x}}, Y^{\tilde{x}}, Z^{\tilde{x}})$ of (4.1), such that $X^{\tilde{x}}(\omega)$ and $Y^{\tilde{x}}(\omega)$ are continuously differentiable in $\tilde{x} \in \mathbb{R}^{(n+d) \times 1}$ for almost all $\omega \in \Omega$. Furthermore there exist processes $\frac{\partial}{\partial x} Z^{x,m}, \frac{\partial}{\partial m} Z^{x,m} \in L^2(d\langle M, M \rangle \otimes d\mathbb{P})$ such that the derivatives $(U_k^x, V_{ik}^x) := (\frac{\partial}{\partial x_k} Y^{x,m}, \frac{\partial}{\partial x_k} Z^{(i),x,m})$, $i = 1, \dots, d$, $k = 1, \dots, n$, and $(U_k^m, V_{ik}^m) := (\frac{\partial}{\partial m_k} Y^{x,m}, \frac{\partial}{\partial m_k} Z^{(i),x,m})$, $i, k = 1, \dots, d$, solve the following BSDE for $t \in [0, T]$*

$$\begin{aligned}
U_{kt}^x &= \sum_{j=1}^n \partial_j F(X_T^{x,m}, M_T^m) D_{jkT}^x - \sum_{\alpha=1}^d \int_t^T V_{i\alpha s}^x dM_s^{(\alpha)} \\
&\quad + \sum_{j=1}^n \int_t^T \partial_{1+j} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m} q_s^*) D_{jks}^x dC_s \\
&\quad + \int_t^T \partial_{1+n+d+1} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m} q_s^*) U_{ks}^x dC_s \\
&\quad + \sum_{j=1}^n \int_t^T \partial_{1+n+d+1+j} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m} q_s^*) \frac{\partial}{\partial x_k} q_{jks} Z^{(j),x,m} dC_s,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
U_{kt}^m &= \sum_{j=1}^d \partial_{n+j} F(X_T^{x,m}, M_T^m) D_{jkT}^m - \sum_{\alpha=1}^d \int_t^T V_{i\alpha s}^m dM_s^{(\alpha)} \\
&+ \sum_{j=1}^n \int_t^T \partial_{1+j} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m}) D_{jks}^m dC_s \\
&+ \int_t^T \partial_{1+n+k} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m} q_s^*) dC_s \\
&+ \int_t^T \partial_{1+n+d+1} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m} q_s^*) U_{ks}^m dC_s \\
&+ \sum_{j=1}^n \int_t^T \partial_{1+n+d+1+j} f(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, Z_s^{x,m} q_s^*) q_{jks} V_{jks}^m dC_s.
\end{aligned}$$

Proof. From 4.4 and Kolmogorov's Lemma (see Theorem 73, Chapter IV in [22]) we deduce that there exists a family of solutions $(Y^{\tilde{x}})$ of (4.1) which is continuously differentiable in \tilde{x} for almost all $\omega \in \Omega$. Finally from equation (4.10) taking $h \rightarrow 0$ the BSDE follows. \square

4.2 Differentiability of FBSDE

Now we come back to the initial FBSDE (2.1)-(2.2). In order to obtain the differentiability of this system we require the following additional assumption:

(MRP) There exists a constant C such that $\langle M, M \rangle_T \leq C$, \mathbb{P} -a.s.. In addition, there exists a continuous L^2 -martingale $N := (N_t)_{t \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ strongly orthogonal to M ($\langle M^i, N \rangle = 0$, $i = 1, \dots, d$) with $\langle N, N \rangle_T \leq C$, \mathbb{P} -a.s. such that the martingale L part of the solution of (2.2) can be written as $L_t = \int_0^t U_s dN_s$, $t \in [0, T]$ for a predictable square integrable stochastic process U .

The presence of the additional bracket $\langle L, L \rangle$ in the BSDE prevents us from applying the known techniques for differentiability in the Brownian case as shown in [2, 3, 5]. Nevertheless, under (MRP) we can show that the BSDE in (2.2) can be written as

$$\begin{aligned}
Y_t^{x,m} &= F(X_T^{x,m}, M_T^m) - \int_t^T Z_r^{x,m} dM_r - \int_t^T U_r^{x,m} dN_r \\
&+ \int_t^T h(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*, U_r^{x,m}) d\tilde{C}_r,
\end{aligned} \tag{4.12}$$

$t \in [0, T]$, where \tilde{C} and h are defined as in section A.1. Due to hypotheses (MRP) and the orthogonality of the martingales L and M , the representation of L as $L = U \cdot N$ where U is a predictable square integrable stochastic process is obtained. So the solution (Y, Z, L) of the backward part (2.2) becomes (Y, Z, U) in (4.12). The bracket $\langle L, L \rangle$ is then a component of the new generator h , which is quadratic in U . We refer to the Appendix A.1, where a discussion of the technical aspects is given. Now we can write the system (2.1)-(2.2) as

$$X_t^{x,m} = x + \int_0^t \tilde{\sigma}(s, X_s^{t,x,m}, M_s^{t,m}) d\tilde{M}_s + \int_0^t \tilde{b}(s, X_s^{x,m}, M_s^m) d\tilde{C}_s, \tag{4.13}$$

$$Y_t^{x,m} = F(X_T^{x,m}, M_T^m) - \int_t^T \tilde{Z}_s^{x,m} d\tilde{M}_s + \int_t^T h(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, \tilde{Z}_s^{x,m} \tilde{q}_s^*) d\tilde{C}_s,$$

$t \in [0, T]$, where \tilde{M} , \tilde{q} , \tilde{Z} , are defined as in Section A.1 and $\tilde{\sigma} := (\sigma \ 0)$, $\tilde{b} := b \times \varphi_1$ where φ_1 is a bounded predictable process defined in Appendix A.1. A solution $(X^{x,m}, Y^{x,m}, \tilde{Z}^{x,m}) \in \mathcal{S}^p \times \mathcal{S}^\infty \times L^2(d\tilde{M}, \tilde{M}) \otimes d\mathbb{P}$ of this system exists for σ , b satisfying (HSDE) and F , h satisfying (H1)-(H3). Therefore we obtain the following result, whose proof follows from Proposition 4.5.

Theorem 4.6. *Assume that M be a strong Markov process and that f and F in (2.2) satisfy (H1)-(H3) and (D1)-(D4). Under the assumption (MRP) there exists a solution $(X^{\tilde{x}}, Y^{\tilde{x}}, \tilde{Z}^{\tilde{x}})$ of (2.1)-(2.2), such that $X^{\tilde{x}}(\omega)$ and $Y^{\tilde{x}}(\omega)$ are continuously differentiable in $\tilde{x} \in \mathbb{R}^{(n+d) \times 1}$ for almost all $\omega \in \Omega$ (we recall that \tilde{x} stands for (x, m)).*

Proof. Note that the processes $Y^{\tilde{x}}$ of the transformed BSDE (4.12) and of the original BSDE (2.2) coincide. In addition the process $(Z^{\tilde{x}}, L^{\tilde{x}})$ in (2.2) and the processes $\tilde{Z}^{\tilde{x}}$ in (4.12) are related as follows: $\tilde{Z}^{\tilde{x}} = (Z^{\tilde{x}}, U^{\tilde{x}})$ with $L^{\tilde{x}} = \int_0^\cdot U_r^{\tilde{x}} dN_r$ and N is the process coming from (MRP). The definition of the driver h of the BSDE (4.12) (see Appendix A.1), the fact that f and F in (2.2) satisfy and (H1)-(H3) and (D1)-(D4), imply that F and h satisfy also the assumptions (H1)-(H3) and (D1)-(D4). Thus $Y^{\tilde{x}}$ and $Z^{\tilde{x}}$ are continuously differentiable in \tilde{x} by Proposition 4.5 which concludes the proof. \square

Proposition 4.7. *Assume that M is a strong Markov process and that f and F in (2.2) satisfy (H1)-(H3) and (D1)-(D4). From Theorem 3.4 there exists a deterministic function u such that $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$, $s \in [t, T]$. Under the assumption (MRP) we have that*

- i) $x \mapsto u(t, x, m) \in \mathcal{C}^1(\mathbb{R}^{n \times 1})$, $(t, m) \in [0, T] \times \mathbb{R}^{d \times 1}$,
- ii) $m \mapsto u(t, x, m) \in \mathcal{C}^1(\mathbb{R}^{d \times 1})$, $(t, x) \in [0, T] \times \mathbb{R}^{n \times 1}$,
- iii) there exists two constants ζ_1, ζ_2 depending only on $\|F\|_\infty$, a and b of assumption (H2) such that

$$\zeta_1 \leq u(t, x, m) \leq \zeta_2$$

for all $(t, x, m) \in [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1}$,

- iv) the maps

$$(t, x, m) \mapsto \partial_i u(t, x, m)$$

are continuous for $i = 2, 3$.

Proof.

- i) Fix (t, m) in $[0, T] \times \mathbb{R}^{d \times 1}$. As already mentioned, $Y_t^{t,x,m}$ is deterministic and $u(t, x, m) = Y_t^{t,x,m}$. By differentiability of $Y^{t,x,m}$ with respect to x (Theorem 4.6), we obtain that $x \mapsto u(t, x, m)$ belongs to $\mathcal{C}^1(\mathbb{R}^{n \times 1})$.

- ii) The proof is similar to i).

- iii) Let $(t, x, m) \in [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1}$. By [18, Lemma 3.1 (i)] there exists ζ_1, ζ_2 depending only on $|F|_\infty$, a and b such that $\zeta_1 \leq Y_s^{t,x,m} \leq \zeta_2$ for all s in $[t, T]$, \mathbb{P} -a.s.. Thus $\zeta_1 \leq u(t, x, m) = Y_t^{t,x,m} \leq \zeta_2$. Since the constants ζ_1 and ζ_2 do not depend on (t, x, m) the claim is proved.
- iv) For better readability we prove this claim for $d = n = 1$. The moredimensional case is a straitforward extension of the following computations where we adapt [16, Theorem 3.1]. From (3.8) we know that $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$ and hence $Y_t^{t,x,m} = u(t, x, m)$. In the following we use the representation (4.13) of the forward backward system, that is we use the transformed FBSDE. Then by definition of the driver h (see Appendix A.1) the properties of f carry over to h . Thus by Proposition 4.5, the processes $(\nabla_x Y^{t,x,m}, \nabla_x \tilde{Z}^{t,x,m})$ satisfy the following BSDE

$$\begin{aligned} \nabla_x Y_s^{t,x,m} &= \nabla_x F(X_T^{t,x,m}, M_T^{t,m}) \nabla_x X_T^{t,x,m} - \int_s^T \nabla_x \tilde{Z}_r^{t,x,m} d\tilde{M}_r \\ &+ \int_s^T \left(\partial_2 h(r, \Theta_r(t, x, m)) \nabla_x X_r^{t,x,m} \right. \\ &\quad \left. + \partial_4 h(r, \Theta_r(t, x, m)) \nabla_x Y_r^{t,x,m} + \partial_5 h(r, \Theta_r(t, x, m)) \tilde{q}_r \nabla_x \tilde{Z}_r^{t,x,m} \right) d\tilde{C}_s. \end{aligned}$$

Thus putting $s = t$ in the above expression and taking the expectation we get

$$\begin{aligned} &\partial_x u(t, x, m) \\ &= \mathbb{E} \left[\nabla_x F(X_T^{t,x,m}, M_T^{t,m}) \nabla_x X_T^{t,x,m} + \int_t^T \left(\partial_2 h(s, \Theta_s(t, x, m)) \nabla_x X_s^{t,x,m} \right. \right. \\ &\quad \left. \left. + \partial_4 h(s, \Theta_s(t, x, m)) \nabla_x Y_s^{t,x,m} + \partial_5 h(s, \Theta_s(t, x, m)) \tilde{q}_s \nabla_x \tilde{Z}_s^{t,x,m} \right) d\tilde{C}_s \right]. \end{aligned}$$

Here we have used $\Theta_s(t, x, \tilde{m}) := (X_s^{t,x,m}, \tilde{M}_s^{t,m}, Y_s^{t,x,m}, \tilde{Z}_s^{t,x,m} \tilde{q}_s)$. Let us fix (t_1, x_1, m_1) and (t_2, x_2, m_2) with $t_1 < t_2$ and denote $\Theta_s^1 := \Theta_s^1(t_1, x_1, m_1)$ and $\Theta_s^2 := \Theta_s^2(t_2, x_2, m_2)$. We write $X^1 := X^{t_1, x_1, m_1}$ and analogously X^2, Y^1, Y^2 , etc. Furthermore we define $\Delta_{1,2} \varphi(s) := \varphi(s, \Theta_s^1) - \varphi(s, \Theta_s^2)$ for any function φ with values in \mathbb{R} . With a positive constant c changing from line to line, we have that

$$\begin{aligned} &|\partial_x u(t_1, x_1, m_1) - \partial_x u(t_2, x_2, m_2)| \\ &\leq \mathbb{E} \left[\nabla_x F(X_T^1, M_T^1) \nabla_x X_T^1 - \nabla_x F(X_T^2, M_T^2) \nabla_x X_T^2 \right] \\ &+ \mathbb{E} \left[\int_{t_1}^{t_2} |\partial_2 h(s, \Theta_s^1)| |\nabla_x X_s^1| + |\partial_4 h(s, \Theta_s^1)| |\nabla_x Y_s^1| + |\partial_5 h(s, \Theta_s^1)| |\tilde{q}_s \nabla_x \tilde{Z}_s^1| d\tilde{C}_s \right] \\ &+ \mathbb{E} \left[\int_{t_2}^T |\Delta_{1,2} \partial_2 h(s)| |\nabla_x X_s^1| + |\Delta_{1,2} \partial_4 h(s)| |\nabla_x Y_s^1| + |\Delta_{1,2} \partial_5 h(s)| |\tilde{q}_s \nabla_x \tilde{Z}_s^1| d\tilde{C}_s \right] \\ &+ \mathbb{E} \left[\int_{t_2}^T |\partial_2 h(s, \Theta_s^2)| |\nabla_x X_s^1 - \nabla_x X_s^2| + |\partial_4 h(s, \Theta_s^2)| |\nabla_x Y_s^1 - \nabla_x Y_s^2| d\tilde{C}_s \right] \\ &+ \mathbb{E} \left[\int_{t_2}^T |\partial_5 h(s, \Theta_s^2)| |\tilde{q}_s (\nabla_x \tilde{Z}_s^1 - \nabla_x \tilde{Z}_s^2)| d\tilde{C}_s \right] \\ &\leq \mathbb{E} \left[|\nabla_x F(X_T^1, M_T^1) - \nabla_x F(X_T^2, M_T^2)| |\nabla_x X_T^1| + |\nabla_x F(X_T^2, M_T^2)| |\nabla_x X_T^1 - \nabla_x X_T^2| \right] \end{aligned}$$

$$\begin{aligned}
& + c\mathbb{E} \left[\int_{t_1}^{t_2} \left(|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^1| \right) \left(|\nabla_x X_s^1| + |\nabla_x Y_s^1| + |\tilde{q}_s \nabla_x \tilde{Z}_s^1| \right) d\tilde{C}_s \right] \\
& + c\mathbb{E} \left[\int_{t_2}^T \left(|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^1| + |\tilde{q}_s \tilde{Z}_s^2| \right) (|X_s^1 - X_s^2| + |M_s^1 - M_s^2| + |Y_s^1 - Y_s^2|) \right. \\
& \quad \left. \times (|\nabla_x X_s^1| + |\nabla_x Y_s^1|) d\tilde{C}_s \right] \\
& + c\mathbb{E} \left[\int_{t_2}^T \left(|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^1| + |\tilde{q}_s \tilde{Z}_s^2| \right) |\tilde{q}_s (\tilde{Z}_s^1 - \tilde{Z}_s^2)| (|\nabla_x X_s^1| + |\nabla_x Y_s^1|) d\tilde{C}_s \right] \\
& + c\mathbb{E} \left[\int_{t_2}^T |\tilde{q}_s \nabla_x \tilde{Z}_s^1| (|X_s^1 - X_s^2| + |M_s^1 - M_s^2| + |Y_s^1 - Y_s^2|) d\tilde{C}_s \right] \\
& + c\mathbb{E} \left[\int_{t_2}^T |\tilde{q}_s \nabla_x \tilde{Z}_s^1| |\tilde{q}_s (\tilde{Z}_s^1 - \tilde{Z}_s^2)| d\tilde{C}_s \right] \\
& + c\mathbb{E} \left[\int_{t_2}^T \left(|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^2| \right) \right. \\
& \quad \left. \times \left(|\nabla_x X_s^1 - \nabla_x X_s^2| + |\nabla_x Y_s^1 - \nabla_x Y_s^2| + |\tilde{q}_s (\nabla_x \tilde{Z}_s^1 - \nabla_x \tilde{Z}_s^2)| \right) d\tilde{C}_s \right] \\
& =: \sum_{i=1}^7 T_i
\end{aligned}$$

where we have used the assumptions (D3) and (D4) in the last inequality. Recall that (t_1, x_1, m_1) is fixed and $t_2 > t_1$. With (4.4) and (4.5) we see

$$\lim_{t_2 \rightarrow t_1; x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_1 = \lim_{x_2 \rightarrow x_1} T_1 = 0.$$

By the monotone convergence theorem we deduce for the second term

$$\lim_{t_2 \rightarrow t_1; x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_2 = \lim_{t_2 \rightarrow t_1} T_2 = 0.$$

We now deal with T_3

$$\begin{aligned}
T_3 & \leq c\mathbb{E} \left[\sup_{s \in [0, T]} (|\nabla_x X_s^1| + |\nabla_x Y_s^1|) \int_{t_2}^T (|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^1| + |\tilde{q}_s \tilde{Z}_s^2|) \right. \\
& \quad \left. \times (|X_s^1 - X_s^2| + |M_s^1 - M_s^2| + |Y_s^1 - Y_s^2|) d\tilde{C}_s \right] \\
& \leq c\mathbb{E} \left[\sup_{s \in [0, T]} (|\nabla_x X_s^1| + |\nabla_x Y_s^1|)^2 \right]^{1/2} \\
& \quad \times \mathbb{E} \left[\left(\int_{t_2}^T (|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^1| + |\tilde{q}_s \tilde{Z}_s^2|) (|X_s^1 - X_s^2| + |M_s^1 - M_s^2| + |Y_s^1 - Y_s^2|) d\tilde{C}_s \right)^2 \right]^{1/2} \\
& \leq c\mathbb{E} \left[\int_{t_2}^T (|\tilde{q}_s \theta_s| + |\tilde{q}_s \tilde{Z}_s^1| + |\tilde{q}_s \tilde{Z}_s^2|)^2 d\tilde{C}_s \int_{t_2}^T (|X_s^1 - X_s^2| + |M_s^1 - M_s^2| + |Y_s^1 - Y_s^2|)^2 d\tilde{C}_s \right]^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq c\mathbb{E}\left[\left(\int_{t_2}^T (|\tilde{q}_s\theta_s| + |\tilde{q}_s\tilde{Z}_s^1| + |\tilde{q}_s\tilde{Z}_s^2|)^2 d\tilde{C}_s\right)^2\right]^{\frac{1}{4}} \\
&\quad \times \mathbb{E}\left[\left(\int_{t_2}^T (|X_s^1 - X_s^2| + |M_s^1 - M_s^2| + |Y_s^1 - Y_s^2|)^2 d\tilde{C}_s\right)^2\right]^{\frac{1}{4}} \\
&\leq c(\tilde{c} + (|x_2|^2 + |m_2|^2)^{p_1}(|x_2 - x_1|^2 + |m_2 - m_1|^2)^{p_2}),
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Here p_1, p_2 are two positive numbers given by the a priori estimates Lemma A.1 and \tilde{c} is a positive constant. Thus we conclude

$$\lim_{t_2 \rightarrow t_1; x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_3 = \lim_{x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_3 = 0.$$

Similarly one shows

$$\lim_{t_2 \rightarrow t_1; x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_5 = 0.$$

We now estimate T_4 and T_7 but we give the details only for T_4 , since those for T_7 follow the same lines. Applying the Cauchy-Schwarz inequality again we get

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_2}^T (|\tilde{q}_s\theta_s| + |\tilde{q}_s\tilde{Z}_s^1| + |\tilde{q}_s\tilde{Z}_s^2|)|\tilde{q}_s(\tilde{Z}_s^1 - \tilde{Z}_s^2)|(|\nabla_x X_s^1| + |\nabla_x Y_s^1|)d\tilde{C}_s\right] \\
&\leq \mathbb{E}\left[\sup_{s \in [0, T]} (|\nabla_x X_s^1| + |\nabla_x Y_s^1|) \int_{t_2}^T (|\tilde{q}_s\theta_s| + |\tilde{q}_s\tilde{Z}_s^1| + |\tilde{q}_s\tilde{Z}_s^2|)|\tilde{q}_s(\tilde{Z}_s^1 - \tilde{Z}_s^2)|d\tilde{C}_s\right] \\
&\leq \mathbb{E}\left[\sup_{s \in [0, T]} (|\nabla_x X_s^1| + |\nabla_x Y_s^1|)^2\right]^{1/2} \\
&\quad \times \mathbb{E}\left[\left(\int_{t_2}^T (|\tilde{q}_s\theta_s| + |\tilde{q}_s\tilde{Z}_s^1| + |\tilde{q}_s\tilde{Z}_s^2|)|\tilde{q}_s(\tilde{Z}_s^1 - \tilde{Z}_s^2)|d\tilde{C}_s\right)^2\right]^{1/2} \\
&\leq c\mathbb{E}\left[\int_{t_2}^T (|\tilde{q}_s\theta_s| + |\tilde{q}_s\tilde{Z}_s^1| + |\tilde{q}_s\tilde{Z}_s^2|)^2 d\tilde{C}_s \int_{t_2}^T |\tilde{q}_s(\tilde{Z}_s^1 - \tilde{Z}_s^2)|^2 d\tilde{C}_s\right]^{1/2} \\
&\leq c\mathbb{E}\left[\left(\int_{t_2}^T (|\tilde{q}_s\theta_s| + |\tilde{q}_s\tilde{Z}_s^1| + |\tilde{q}_s\tilde{Z}_s^2|)^2 dC_s\right)^2\right]^{\frac{1}{4}} \mathbb{E}\left[\left(\int_{t_2}^T |\tilde{q}_s(\tilde{Z}_s^1 - \tilde{Z}_s^2)|^2 d\tilde{C}_s\right)^2\right]^{\frac{1}{4}} \\
&\leq c(\tilde{c} + (|x_2|^2 + |m_2|^2)^{p_1}(|x_2 - x_1|^2 + |m_2 - m_1|^2)^{p_2}).
\end{aligned}$$

Here, as before, p_1, p_2 are two positive numbers given by the a priori estimates Lemma A.1 and \tilde{c} is a positive constant. This leads to

$$\lim_{t_2 \rightarrow t_1; x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_4 = \lim_{x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_4 = 0.$$

Finally we consider the term T_6

$$\mathbb{E}\left[\int_{t_2}^T |\tilde{q}_s \nabla_x \tilde{Z}_s^1| |\tilde{q}_s(\tilde{Z}_s^1 - \tilde{Z}_s^2)| d\tilde{C}_s\right]$$

$$\begin{aligned}
&\leq \mathbb{E} \left[\left(\int_{t_2}^T |\tilde{q}_s \nabla_x \tilde{Z}_s^1|^2 d\tilde{C}_s \right)^{1/2} \left(\int_{t_2}^T |\tilde{q}_s (\tilde{Z}_s^1 - \tilde{Z}_s^2)|^2 d\tilde{C}_s \right)^{1/2} \right] \\
&\leq c \mathbb{E} \left[\int_{t_2}^T |\tilde{q}_s (\tilde{Z}_s^1 - \tilde{Z}_s^2)|^2 d\tilde{C}_s \right]^{1/2} \\
&\leq c(|x_2 - x_1|^2 + |m_2 - m_1|^2)^p,
\end{aligned}$$

where the positive constant p is given by the a priori estimates Lemma A.1. Thus we have

$$\lim_{t_2 \rightarrow t_1; x_2 \rightarrow x_1; m_2 \rightarrow m_1} T_6 = 0.$$

The same methodology shows that for fixed (t_2, x_2, m_2)

$$\lim_{t_1 \rightarrow t_2; x_1 \rightarrow x_2; m_1 \rightarrow m_2} |\partial_x u(t_1, x_1, m_1) - \partial_x u(t_2, x_2, m_2)| = 0.$$

Similarly, we can show that $\partial_m u$ is continuous in (t, x, m) .

□

Example of stochastic basis where the condition (MRP) is satisfied

Let $(B^1, B^2) := (B_s^1, B_s^2)_{s \in [0, T]}$ be a two dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a terminal time $0 < T < \infty$ and with B^1 and B^2 being independent. We denote by $(\mathcal{F}_t)_{t \in [0, T]}$ the filtration generated by (B^1, B^2) . Then the process $M := (B_t^1)_{t \in [0, T]}$ is a continuous martingale with respect to $(\mathcal{F}_t)_{t \in [0, T]}$ and it is a $(\mathcal{F}_t)_{t \in [0, T]}$ -strong Markov process. Let $N = (B_t^2)_{t \in [0, T]}$. The martingale representation property for (B^1, B^2) and the strong orthogonality between B^1 and B^2 entail that the pair (M, N) satisfies the property (MRP) introduced in Section 4.2.

5 Representation formula

In this Section we provide the representation formula (1.3) which generalizes the one obtained in [2, 3], where M is a Brownian motion. We recall that in the Gaussian setting the proof of this formula is based on the representation of the stochastic process Z as the trace of the Malliavin derivative of Y . In the general martingale setting of this paper Malliavin's calculus is not available, therefore we propose a new proof based on stochastic calculus techniques. We also stress that the last term in formula (1.3) vanishes if we assume that M has independent increments, σ and b do not depend on M in (2.1) and that the driver f in (2.2) is independent of M .

We present the main result of this paper. We stress that this result does not rely on the assumption (MRP) made in Section 4.2 since only the regularity of the deterministic function u where $Y = u(\cdot, X, M)$ is needed.

Theorem 5.1. Assume that M is a Markov process. Assume that (HSDE), (H1)-(H3) are in force for the FBSDE (2.1)-(2.2). Then by Theorem 3.4, there exists a deterministic function u such that $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$, $s \in [t, T]$. Assume in addition that u satisfies:

- i) $x \mapsto u(t, x, m) \in \mathcal{C}^1(\mathbb{R}^{n \times 1})$, $(t, m) \in [0, T] \times \mathbb{R}^{d \times 1}$,
- ii) $m \mapsto u(t, x, m) \in \mathcal{C}^1(\mathbb{R}^{d \times 1})$, $(t, x) \in [0, T] \times \mathbb{R}^{n \times 1}$,
- iii) there exist two constants ζ_1, ζ_2 depending only on $\|F\|_\infty$, a and b of assumption (H2) such that
$$\zeta_1 \leq u(t, x, m) \leq \zeta_2, \quad \forall (t, x, m) \in [0, T] \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{d \times 1},$$
- iv) the maps
$$(t, x, m) \mapsto \partial_i u(t, x, m) \quad \text{are continuous for } i = 2, 3$$

then for all $s \in [t, T]$ we have ν -a.e.

$$Z_s^{t,x,m} = \partial_2 u(s, X_s^{t,x,m}, M_s^{t,m}) \sigma(s, X_s^{t,x,m}, M_s^{t,m}) + \partial_3 u(s, X_s^{t,x,m}, M_s^{t,m}). \quad (5.1)$$

Remark 5.2. i) An interesting particular case of Theorem 5.1 is given when X and M are as in Proposition 3.1 ii) and when f in (3.7) does not depend on M . In this situation equation (5.1) becomes:

$$Z_s^{t,x} = \partial_2 u(s, X_s^{t,x}) \sigma(s, X_s^{t,x}), \quad \nu - a.e.,$$

which coincides with the representation formula derived in [2, 3] when M is a standard Brownian motion.

- ii) One may be interested in knowing when u in Theorem 5.1 does not depend trivially on M , i.e. when the third term in (5.1) does not vanish. This is related to the Markov property given for Y and we provide in Section A.3 an explicit example where u depends non-trivially on M .

Proof of Theorem 5.1. Fix s in $[t, T]$. For simplicity of notations we drop the superscript (t, x, m) . We briefly explain the idea of the proof. Assume that the function u introduced above is in $\mathcal{C}^{1,2}$ that is continuously differentiable in time and twice continuously differentiable in (x, m) . Then an application of Itô's formula gives that

$$\langle Y, M \rangle_s = \langle u(\cdot, X, M), M \rangle_s = \int_t^s [\partial_2 u(r, X_r, M_r) \sigma(r, X_r, M_r) + \partial_3 u(r, X_r, M_r)] d\langle M, M \rangle_r, \quad (5.2)$$

where we denote by $\langle u(\cdot, X_s, M_s), M \rangle_s$ the covariation vector

$$\left(\langle u(\cdot, X, M), M^{(1)} \rangle_s, \dots, \langle u(\cdot, X, M), M^{(d)} \rangle_s \right).$$

Then since (Y, Z) is solution of (2.2) we have that

$$\langle Y, M \rangle_s = \int_t^s Z_r d\langle M, M \rangle_r, \quad s \in [t, T]. \quad (5.3)$$

The conclusion of the theorem then follows from the fact that $Y_s = u(s, X_s, M_s)$, $s \in [t, T]$, and from relations (5.2) and (5.3). However we have assumed the function u to be much more regular than what it is and so we have to prove the relation (5.2) for u being only one time differentiable in (x, m) . The rest of the proof is devoted to this fact. For this we compute "directly" the quadratic variation between $u(\cdot, X, M)$ and M .

Fix $i \in \{1, \dots, d\}$. Let $r \geq 1$ and $\pi^{(r)} := \{t_j^{(r)}, j = 1, \dots, r\}$ be a partition of $[t, T]$ whose mesh size $|\pi^{(r)}|$ tends to zero as r goes to infinity with $t_0^{(r)} = t$ and $t_r^{(r)} = T$ such that

$$\lim_{r \rightarrow \infty} \sup_{t \leq s \leq T} \left| \langle u(\cdot, X, M), M^{(i)} \rangle_s - \sum_{j=0}^{\varphi_s-1} \left(u(t_{j+1}^{(r)}, X_{t_{j+1}^{(r)}}, M_{t_{j+1}^{(r)}}) - u(t_j^{(r)}, X_{t_j^{(r)}}, M_{t_j^{(r)}}) \right) \Delta_j M^{(i)} \right| = 0$$

where the limit is understood in probability with respect to \mathbb{P} , $\Delta_j M$ denotes the increments of the stochastic process M on $[t_j^{(r)}, t_{j+1}^{(r)}]$ and $\varphi_s^{(r)}$ is such that $\varphi_s^{(r)} = j$ with $t_j^{(r)} \leq s < t_{j+1}^{(r)}$. For simplicity of notation the superscript (r) will be omitted. In addition, up to a subsequence we can assume that convergence above is almost sure with respect to \mathbb{P} . We have that

$$\begin{aligned} & \langle u(\cdot, X, M), M^{(i)} \rangle_s \\ &= \lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_j}, M_{t_j})) \Delta_j M^{(i)} \\ &= \lim_{r \rightarrow \infty} \left[\sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_j}, M_{t_j}) - u(t_j, X_{t_j}, M_{t_j})) \Delta_j M^{(i)} \right. \\ & \quad \left. + \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_{j+1}, X_{t_j}, M_{t_j})) \Delta_j M^{(i)} \right] \\ &=: \lim_{r \rightarrow \infty} [S_{s,r,1}^{(i)} + S_{s,r,2}^{(i)}]. \end{aligned} \tag{5.4}$$

We treat the two parts separately. First assume that the second term converges, more precisely that the relation (5.5) below holds:

$$\begin{aligned} & \lim_{r \rightarrow \infty} \sup_{t \leq s \leq T} \left| S_{s,r,2}^{(i)} - \left(\int_t^s [\partial_2 u(r, X_r, M_r) \sigma(r, X_r, M_r) + \partial_3 u(r, X_r, M_r)] d\langle M, M \rangle_r \right)^{(i)} \right| \\ &= 0, \mathbb{P} - a.s. \end{aligned} \tag{5.5}$$

It then follows by relations (5.3) and (5.4) that

$$\lim_{r \rightarrow \infty} \sup_{t \leq s \leq T} |S_{s,r,1}^{(i)} - P_s| = 0, \mathbb{P} - a.s.$$

with

$$P_s := \left(\int_t^s Z_a - \partial_2 u(a, X_a, M_a) \sigma(a, X_a, M_a) - \partial_3 u(a, X_a, M_a) d\langle M, M \rangle_a \right)^{(i)}, \quad s \in [t, T].$$

We will show that P is \mathbb{P} -a.s. identically equal to zero. Since u is not differentiable in time, one can a priori not say how the sum $S_{s,r,1}^{(i)}$ behaves asymptotically. However, we know that it converges and that its limit is absolutely continuous with respect to $d\langle M, M \rangle$. Heuristically, this means that each term of the form $u(t_{j+1}, X_{t_j}, M_{t_j}) - u(t_j, X_{t_j}, M_{t_j})$ behaves like a process times an increment of $\Delta_j M^{(i)}$ which is not possible since u is a deterministic function. We will show that P is a local martingale. Since by definition, it is a finite variation process we will have $P = 0$. We first make the following assumption that we will relax later. Assume that

$$\mathbb{E}[|P_s|] < \infty, \quad \forall s \in [t, T]. \quad (5.6)$$

Now fix $t \leq s_1 \leq s_2 \leq T$. For a point t_j in the subdivision considered above we define $\delta_j u := u(t_{j+1}, X_{t_j}, M_{t_j}) - u(t_j, X_{t_j}, M_{t_j})$. We have that

$$\begin{aligned} \mathbb{E}[P_{s_2} | \mathcal{F}_{s_1}] &= \mathbb{E} \left[\lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} | \mathcal{F}_{s_1} \right] \\ &= \mathbb{E} \left[\lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} + (M_{s_2} - M_{\varphi_{s_2}}) | \mathcal{F}_{s_1} \right] \end{aligned} \quad (5.7)$$

since by continuity of the martingale M , $\lim_{r \rightarrow \infty} M_{s_2} - M_{\varphi_{s_2}} = 0$, \mathbb{P} -a.s. (recall that φ_{s_2} tends to s_2 when r goes to infinity). In addition since the function u is bounded (by Proposition 4.7 iii), the sequence $\left(\sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} + (M_{s_2} - M_{\varphi_{s_2}}) \right)_r$ is uniformly bounded. Indeed we have that

$$\begin{aligned} &\mathbb{E} \left[\left| \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} + (M_{s_2} - M_{\varphi_{s_2}}) \right|^2 \right] \\ &= \sum_{j=0}^{\varphi_{s_2}-1} \mathbb{E} \left[|\delta_j u|^2 |\Delta_j M^{(i)}|^2 \right] + \mathbb{E}[(M_{s_2} - M_{\varphi_{s_2}})^2] \\ &\leq c \left(\sum_{j=0}^{\varphi_{s_2}-1} \mathbb{E} \left[|M_{t_{j+1}}^{(i)}|^2 \right] - \mathbb{E} \left[|M_{t_j}^{(i)}|^2 \right] + \mathbb{E}[|M_{s_2}|^2] - \mathbb{E}[|M_{\varphi_{s_2}}|^2] \right) \\ &= c(\mathbb{E}[|M_{s_2}|^2] - m) \end{aligned}$$

thus $\sup_r \mathbb{E} \left[\left| \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} + (M_{s_2} - M_{\varphi_{s_2}}) \right|^2 \right] \leq c(\mathbb{E}[|M_{s_2}|^2] - m) < \infty$. Using the Lebesgue dominated convergence Theorem in (5.7) we get

$$\begin{aligned} \mathbb{E}[P_{s_2} | \mathcal{F}_{s_1}] &= \lim_{r \rightarrow \infty} \mathbb{E} \left[\sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} + (M_{s_2}^{(i)} - M_{\varphi_{s_2}}^{(i)}) | \mathcal{F}_{s_1} \right] \\ &= \lim_{r \rightarrow \infty} \left(\sum_{j=0}^{\varphi_{s_1}-1} \delta_j u \Delta_j M^{(i)} + \mathbb{E} \left[(\delta_{\varphi_{s_1}} u) \Delta_{\varphi_{s_1}} M^{(i)} | \mathcal{F}_{s_1} \right] \right) \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \left[\sum_{j=\varphi_{s_1}+1}^{\varphi_{s_2}-1} \delta_j u \Delta_j M^{(i)} + (M_{s_2}^{(i)} - M_{t_{\varphi_{s_2}}}^{(i)}) | \mathcal{F}_{s_1} \right] \Bigg) \\
& = \lim_{r \rightarrow \infty} \left(\sum_{j=0}^{\varphi_{s_1}-1} \delta_j u \Delta_j M^{(i)} + (\delta_{\varphi_{s_1}} u)(M_{s_1}^{(i)} - M_{t_{\varphi_{s_1}}}^{(i)}) \right) \\
& = P_{s_1}.
\end{aligned}$$

Thus P is a martingale which has (by definition) finite variation, so it has zero quadratic variation and hence

$$P_s = 0, \quad \forall s \in [t, T]$$

which proves

$$\lim_{r \rightarrow \infty} \sup_{t \leq s \leq T} |S_{s,r,1}^{(i)}| = 0, \quad \mathbb{P} - a.s..$$

Now we have to relax the assumption (5.6). Since P is a continuous semimartingale by definition there exists a sequence of stopping times $(T_m)_m$ with $\lim_{m \rightarrow \infty} T_m = T$, \mathbb{P} -a.s. such that $(P_{s \wedge T_m})_{s \in [t, T]}$ is integrable for all $m \geq 1$. Using this localization the previous argument leads to $P_{s \wedge T_m} = 0$ for all $s \in [t, T]$, \mathbb{P} -a.s.. By letting m go to infinity we get

$$\lim_{r \rightarrow \infty} \sup_{t \leq s \leq T} |S_{s,r,1}^{(i)}| = 0, \quad \mathbb{P} - a.s..$$

It remains to show that relation (5.5) holds. Let $s \in [t, T]$. We have that

$$\begin{aligned}
\lim_{r \rightarrow \infty} S_{s,r,2}^{(i)} &= \lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_{j+1}, X_{t_j}, M_{t_j})) \Delta_j M^{(i)} \\
&= \lim_{r \rightarrow \infty} \left[\sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}, M_{t_j}) - u(t_{j+1}, X_{t_j}, M_{t_j})) \Delta_j M^{(i)} \right. \\
&\quad \left. + \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_{j+1}, X_{t_{j+1}}, M_{t_j})) \Delta_j M^{(i)} \right]. \tag{5.8}
\end{aligned}$$

In addition we can write

$$\begin{aligned}
& u(t_{j+1}, X_{t_{j+1}}, M_{t_j}) - u(t_{j+1}, X_{t_j}, M_{t_j}) \\
&= \sum_{k=1}^n \left(u(t_{j+1}, X_{t_j}^{(1)}, \dots, X_{t_j}^{(k-1)}, X_{t_{j+1}}^{(k)}, \dots, X_{t_{j+1}}^{(n)}, M_{t_j}) \right. \\
&\quad \left. - u(t_{j+1}, X_{t_j}^{(1)}, \dots, X_{t_j}^{(k-1)}, X_{t_j}^{(k)}, \dots, X_{t_{j+1}}^{(n)}, M_{t_j}) \right).
\end{aligned}$$

Each term of this sum can be written as

$$\begin{aligned}
& u(t_{j+1}, X_{t_j}^{(1)}, \dots, X_{t_j}^{(k-1)}, X_{t_{j+1}}^{(k)}, \dots, X_{t_{j+1}}^{(n)}, M_{t_j}) - u(t_{j+1}, X_{t_j}^{(1)}, \dots, X_{t_j}^{(k-1)}, X_{t_j}^{(k)}, \dots, X_{t_{j+1}}^{(n)}, M_{t_j}) \\
&= (\partial_2^- u)(\Delta_j X^{(1)}, \dots, \Delta_j X^{(n)})^* \tag{5.9}
\end{aligned}$$

where $\partial_2^- u := \left(\partial_{1+k} u(t_{j+1}, X_{t_j}^{(1)}, \dots, X_{t_j}^{(k-1)}, \bar{X}_{t_j}^{(k)}, X_{t_{j+1}}^{(k+1)}, \dots, X_{t_{j+1}}^{(n)}, M_{t_j}) \right)_{1 \leq k \leq n}$ and $\bar{X}_{t_j}^{(k)}$ is a suitable random point in the interval $[X_{t_j}^{(k)} \wedge X_{t_{j+1}}^{(k)}, X_{t_j}^{(k)} \vee X_{t_{j+1}}^{(k)}]$. Similarly we obtain

$$u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_{j+1}, X_{t_{j+1}}, M_{t_j}) = (\partial_3^- u)(\Delta_j M^{(1)}, \dots, \Delta_j M^{(d)})^* \quad (5.10)$$

with $\partial_3^- u := \left(\partial_{1+n+k} u(t_{j+1}, X_{t_{j+1}}, M_{t_j}^{(1)}, \dots, M_{t_j}^{(k-1)}, \bar{M}_{t_j}^{(k)}, M_{t_{j+1}}^{(k+1)}, \dots, M_{t_{j+1}}^{(d)}) \right)_{1 \leq k \leq d}$. Combining relations (5.8), (5.9) and (5.10) we deduce that

$$\lim_{r \rightarrow \infty} S_{r,2}^{(i)} = \lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_s-1} \left[(\partial_2^- u)(\Delta_j X^{(1)}, \dots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} \right. \quad (5.11)$$

$$\left. + (\partial_3^- u)(\Delta_j M^{(1)}, \dots, \Delta_j M^{(d)})^* \Delta_j M^{(i)} \right] \\ = \lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_s-1} \left[\partial_2 u(t_j, X_{t_j}, M_{t_j})(\Delta_j X^{(1)}, \dots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} \right. \\ \left. + \partial_3 u(t_j, X_{t_j}, M_{t_j})(\Delta_j M^{(1)}, \dots, \Delta_j M^{(d)})^* \Delta_j M^{(i)} + R(i, j, r) \right], \quad (5.12)$$

where $R(i, j, r)$ is defined as

$$R(i, j, r) = ((\partial_2^- u) - \partial_2 u(t_j, X_{t_j}, M_{t_j})) (\Delta_j X^{(1)}, \dots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} \\ + ((\partial_3^- u) - \partial_3 u(t_j, X_{t_j}, M_{t_j})) (\Delta_j M^{(1)}, \dots, \Delta_j M^{(d)})^* \Delta_j M^{(i)}.$$

Since

$$\lim_{r \rightarrow \infty} \sum_{j=0}^{\varphi_s-1} \left[\partial_2 u(t_j, X_{t_j}, M_{t_j})(\Delta_j X^{(1)}, \dots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} \right. \quad (5.13)$$

$$\left. + (\partial_3 u)(\Delta_j M^{(1)}, \dots, \Delta_j M^{(d)})^* \Delta_j M^{(i)} \right] \\ = \left(\int_t^s [\partial_2 u(r, X_r, M_r) \sigma(r, X_r, M_r) + \partial_3 u(r, X_r, M_r)] d\langle M, M \rangle_r \right)^{(i)}, \quad (5.14)$$

relation (5.2) follows from equations (5.11) and (5.13) provided the following equation (5.15) holds:

$$\lim_{r \rightarrow \infty} \left| \sup_{t \leq s \leq T} \sum_{j=0}^{\varphi_s-1} R(i, j, r) \right| = 0. \quad (5.15)$$

We conclude the proof by showing relation (5.15). Let

$$A^{(r)} := \sup_{|s_1 - s_2| \leq |\pi^{(r)}|, a, b \in [s_1, s_2], k=1, \dots, n} \{ |\partial_{1+k} u(s_2, X_a, M_{s_1}) - \partial_{1+k} u(s_1, X_b, M_{s_1})| \} \quad \text{and} \\ B^{(r)} := \sup_{|s_1 - s_2| \leq |\pi^{(r)}|, a, b \in [s_1, s_2], k=1, \dots, d} \{ |\partial_{1+n+k} u(s_2, X_{s_2}, M_a) - \partial_{1+n+k} u(s_1, X_{s_2}, M_b)| \}.$$

For $1 \leq i \leq d$, $r \in \mathbb{N}$ we have for any s in $[t, T]$ that

$$\begin{aligned}
& \sum_{j=0}^{\varphi_s-1} |R(i, j, r)| \\
& \leq cA^{(r)} \sum_{j=0}^{\varphi_s-1} \sum_{k=1}^n |\Delta_j X^{(k)} \Delta_j M^{(i)}| + CB^{(r)} \sum_{j=0}^{\varphi_s-1} \sum_{k=1}^d |\Delta_j M^{(k)} \Delta_j M^{(i)}| \\
& \leq \frac{c}{2} A^{(r)} \sum_{j=0}^{\varphi_s-1} \sum_{k=1}^n \left[|\Delta_j X^{(k)}|^2 + |\Delta_j M^{(i)}|^2 \right] + \frac{c}{2} B^{(r)} \sum_{j=0}^{\varphi_s-1} \sum_{k=1}^d \left[|\Delta_j M^{(k)}|^2 + |\Delta_j M^{(i)}|^2 \right] \\
& \leq \frac{c}{2} A^{(r)} \sum_{k=1}^n \left[\sum_{j_1=1}^{r-1} |\Delta_{j_1} X^{(k)}|^2 + \sum_{j_2=1}^{r-1} |\Delta_{j_2} M^{(i)}|^2 \right] \\
& \quad + \frac{c}{2} B^{(r)} \sum_{k=1}^d \left[\sum_{j_1=1}^{r-1} |\Delta_{j_1} M^{(k)}|^2 + \sum_{j_2=1}^{r-1} |\Delta_{j_2} M^{(i)}|^2 \right].
\end{aligned}$$

Thus

$$\begin{aligned}
& \sup_{t \leq s \leq T} \sum_{j=0}^{\varphi_s-1} |R(i, j, r)| \\
& \leq \frac{c}{2} A^{(r)} \sum_{k=1}^n \left[\sum_{j_1=1}^{r-1} |\Delta_{j_1} X^{(k)}|^2 + \sum_{j_2=1}^{r-1} |\Delta_{j_2} M^{(i)}|^2 \right] + \frac{c}{2} B^{(r)} \sum_{k=1}^d \left[\sum_{j_1=1}^{r-1} |\Delta_{j_1} M^{(k)}|^2 + \sum_{j_2=1}^{r-1} |\Delta_{j_2} M^{(i)}|^2 \right].
\end{aligned}$$

According to Proposition 4.7 iv) we have that

$$\lim_{r \rightarrow \infty} A^{(r)} = \lim_{r \rightarrow \infty} B^{(r)} = 0, \quad \mathbb{P} - a.s.$$

On the other hand

$$\begin{cases} \lim_{r \rightarrow \infty} \sum_{j=1}^{r-1} |\Delta_j X^{(k)}|^2 = \langle X^{(k)}, X^{(k)} \rangle_z, \\ \lim_{r \rightarrow \infty} \sum_{j=1}^{r-1} |\Delta_j M^{(k)}|^2 = \langle M^{(k)}, X^{(k)} \rangle_z, \end{cases}$$

which concludes the proof. \square

As an immediate consequence of Proposition 4.7 and of Theorem 5.1 we get the following corollary.

Corollary 5.3. *Assume that M is a Markov process. Assume that (HSDE), (H1)-(H3) are in force for the FBSDE (2.1)-(2.2). Then by Theorem 3.4, there exists a deterministic function u such that $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$, $s \in [t, T]$. Assume in addition that the assumption (MRP) (see Section 4.2) is in force, then for all $s \in [t, T]$ we have ν -a.e.*

$$Z_s^{t,x,m} = \partial_2 u(s, X_s^{t,x,m}, M_s^{t,m}) \sigma(s, X_s^{t,x,m}, M_s^{t,m}) + \partial_3 u(s, X_s^{t,x,m}, M_s^{t,m}).$$

6 Application to utility based pricing and hedging in incomplete markets

In this section we study the exponential utility based indifference price approach for pricing and hedging insurance related derivatives in incomplete markets. Thereby we will interpret relation (5.1) as a delta hedging formula. Since in the Brownian setting it is shown in [3] that this relation can be expressed as a function of the gradient of the indifference price and correlation coefficients, we only sketch the arguments here. Let us explain how these quantities translate into our local martingale framework with the more complex Markovian structure. Consider an n -dimensional process describing non-tradable risk

$$R_s^{t,r,m} = r + \int_t^s \sigma(u, R_u^{t,r,m}, M_u^{t,m}) dM_u + \int_t^s b(u, R_u^{t,r,m}, M_u^{t,m}) dC_u, \quad s \in [t, T],$$

where $\sigma \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^{n \times 1}$ are measurable functions. An agent aims to price and hedge a derivative of the form $F(R_T^{t,r,m})$, with F being a bounded measurable function. The hedging instrument is a financial market consisting of k risky assets in units of the numeraire that evolve according to the following SDE

$$dS_s = S_s(\beta(s, R_s^{t,r,m}, M_s^{t,m}) dM_s + \alpha(s, R_s^{t,r,m}, M_s^{t,m}) dC_s), \quad s \in [t, T],$$

where the measurable processes α and β take their values in $\mathbb{R}^{k \times 1}$ resp. in $\mathbb{R}^{k \times d}$. Observe that the price processes of tradable assets S are linked to the risk process via the martingale M , its quadratic variation and the functions β and σ . In addition we assume $k \leq d$ in order to exclude arbitrage opportunities. The small agent's preferences are represented through the exponential utility function with risk aversion coefficient $\kappa > 0$, i.e.

$$U(x) = -e^{-\kappa x}, \quad x \in \mathbb{R}.$$

The agent wants to maximize his expected utility by trading in the market. His value function is given by

$$V^F(x, t, r, m) = \sup_{\lambda} \mathbb{E} \left[U(x + \sum_{i=1}^k \int_t^T \lambda_s^{(i)} \frac{dS_s^{(i)}}{S_s^{(i)}} + F(R_T^{t,r,m})) \right],$$

where x is his initial capital and $\lambda^{(i)}$ denotes the momentary value of his portfolio fraction invested in the i -th asset. This optimization problem can be reduced to solving a quadratic BSDE whose generator has been given in [12] for the Brownian case and then extended to our setting in [18]. A way to price and hedge the derivative $F(R_T^{t,r,m})$ is to consider the indifference price $p(t, r, m)$ defined via $V^F(x - p(t, r, m), t, r, m) = V^0(x, t, r, m)$. According to [3] the indifference price can be expressed as $p(t, r, m) = Y^{F,t,r,m} - Y^{0,t,r,m}$, where $(Y^{F,t,r,m}, Z^{F,t,r,m}, L^{F,t,r,m})$ is the solution of the BSDEs

$$\begin{aligned} Y_s^{F,t,r,m} = & F(R_s^{t,r,m}) - \int_s^T Z_u^{F,t,r,m} dM_u + \int_s^T f(u, R_u^{t,r,m}, M_u^{t,m}, Z_u^{F,t,r,m} q_u^*) dC_u \\ & - \int_s^T dL_u^{F,t,r,m} + \frac{\kappa}{2} \int_s^T d\langle L^{F,t,r,m}, L^{F,t,r,m} \rangle_u, \quad t \in [0, T], \end{aligned} \quad (6.1)$$

where the generator f is obtained explicitly through the martingale optimality principle, c.f. [12, 18], and possesses properties covered by the hypotheses of Theorem 5.1. To implement utility indifference, we have to describe the optimal strategies $\hat{\lambda}^F$ and $\hat{\lambda}^0$. In [12] it is shown that $\hat{\lambda}^F \beta(\cdot, R^{t,r,m}, M^{t,m})$ (and $\hat{\lambda}^0 \beta(\cdot, R^{t,r,m}, M^{t,m})$) are given by the projection of a linear function of $Z^{F,t,r} q^*$ (respectively $Z^{0,t,r} q^*$) on the constraint set. Since $R^{t,r,m}$ is not tradable directly, β plays the role of a filter for trading in the market. Due to [3] the optimal strategy to hedge $F(R_T^{t,r,m})$ can be decomposed into a pure trading part $\hat{\lambda}^0$ and the optimal hedge Δ , which is the part of the strategy that replicates the derivative $F(R_T^{t,r,m})$. Using the Markov property given in Theorem 3.4, we see that there exists a deterministic function u^F such that $Y^{F,t,r,m} = u^F(\cdot, R^{t,r,m}, M^{t,m})$. Moreover, the projection mentioned above can be explicitly expressed. Indeed from [3, proof of Theorems 4.2 and 4.4] we have

$$\hat{\lambda}_s^F - \hat{\lambda}_s^0 = (Z_s^{F,t,r,m} - Z_s^{0,t,r,m}) q_s^* \beta^* (\beta \beta^*)^{-1} \beta(s, R_s^{t,r,m}, M_s^{t,m}), \quad s \in [t, T].$$

This leads to

$$\Delta(t, r, m) = (\hat{\lambda}^F - \hat{\lambda}^0) \beta^* (\beta \beta^*)^{-1}(t, r, m) = (Z_t^{F,t,r,m} - Z_t^{0,t,r,m}) q_t^* \beta^* (\beta \beta^*)^{-1}(t, r, m).$$

Using formula (1.3) we derive

$$\Delta(t, r, m) = [\partial_2 p(t, r, m) \sigma(t, r, m) + \partial_3 p(t, r, m)] q_t^* \beta^* (\beta \beta^*)^{-1}(t, r, m). \quad (6.2)$$

We emphasize that, as a consequence of the particular form of the driver f in (6.1), if M has independent increments and the coefficients σ , b , β and α do not depend on M (see Remarks 5.2 *ii*) and *iii*), then relation (6.2) is replaced by

$$\Delta(t, r) = [\partial_2 p(t, r) \sigma(t, r)] q_t^* \beta^* (\beta \beta^*)^{-1}(t, r). \quad (6.3)$$

Finally, note that we obtain formulae (1.3) and (6.2) under condition (MRP) (see Section 4.2). However we believe that this condition is not necessary for deriving (6.2).

Concluding remarks

In this paper we prove the representation formula (1.3) for the control process of a quadratic growth BSDE driven by a continuous local martingale. This can be used for giving an explicit representation of the delta hedge in utility indifference based hedging of insurance derivatives with exponential preferences. We also provide the Markov property and differentiability of the FBSDE (2.1)-(2.2) in the initial state parameter of its forward part. This last property is obtained under an additional assumption (MRP). However, we think that differentiability should hold without this assumption and that different techniques have to be developed for achieving this goal.

Additionally, as already mentioned in this paper, Malliavin's calculus has been used by several authors to recover formula (5.1) in the Brownian framework. Our alternative method is valid in this setting and seems to present advantages in some practical situations. Actually, Malliavin's calculus is known for its efficiency in several topics however it also requires usually more regularity than the problem needs intrinsically. In [1] the authors study the

quadratic hedging problem of contingent claims with basis risk when the hedging instrument and the underlying of the contingent are related via a random correlation process. As given in [1] the hedging strategy is described via a representation formula of the form (5.1) for the control process of the backward part of a FBSDE driven by a Brownian motion. In this case the coefficient of the forward process depends on a correlation process ρ which is itself solution of a Brownian SDE. As explained in a comment in [1, Section 3.4] the use of Malliavin's calculus enforces that the derivatives of the coefficients of the SDE defining ρ have bounded derivatives. This additional regularity is not necessary in our approach and would allow one to consider more examples of correlation processes with only locally Lipschitz bounded derivatives.

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A Appendix

In the first section of this appendix we provide the transformation of a BSDE of the form (2.2) which is needed in Section 4, and give a priori estimates on the solution of the transformed BSDE with respect to its terminal condition and its generator. Then in Appendix A.3 we present an explicit example of situation described in Proposition 3.1 *ii*).

A.1 Transformation of the BSDE (2.2) under (MRP)

We start giving a justification that under (MRP) the BSDE of the form

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T f(s, Y_s, Z_s q_s^*) dC_s - \int_t^T dL_s + \frac{\kappa}{2} \int_t^T d\langle L, L \rangle_s \quad (\text{A.1})$$

can be transformed into a BSDE of the form

$$Y_t = B - \int_t^T \tilde{Z}_s d\tilde{M}_s + \int_t^T h(s, Y_s, \tilde{Z}_s \tilde{q}_s^*) d\tilde{C}_s, \quad (\text{A.2})$$

where for all $s \in [0, T]$

$$\tilde{M}_s := \begin{pmatrix} M_s \\ N_s \end{pmatrix}, \quad \tilde{q}_s := \begin{pmatrix} q_s \sqrt{\varphi_1(s)} & 0 \\ 0 & \sqrt{\varphi_2(s)} \end{pmatrix}, \quad \tilde{C}_s := \arctan \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right),$$

$\tilde{Z}_s := (Z_s, U_s)$, with φ_1 and φ_2 denoting two non-negative positive predictable processes defined below. Let

$$d\mu_s^1 := \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)} \rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2}$$

and

$$d\mu_s^2 := \frac{d\langle N, N \rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2}.$$

For every ω in Ω , the measure $d\mu_t^1(\omega)$ (respectively $d\mu_t^2(\omega)$) is absolutely continuous with respect to $d(\mu_t^1 + \mu_t^2)(\omega)$. Hence, since μ^1 and μ^2 are predictable processes [8, Theorem VI.68 and its remark] imply that there exist two predictable processes φ_1 and φ_2 such that

$$\mu_t^1 = \int_0^t \varphi_1(s) d(\mu + \nu)(s), \quad \mu_t^2 = \int_0^t \varphi_2(s) d(\mu + \nu)(s), \quad \forall t \in [0, T].$$

In addition we can show that $0 \leq \varphi_i(s) \leq 1$ for all s in $[0, T]$ \mathbb{P} -a.s. for $i = 1, 2$. Let $i = 1, 2$. Because φ_i is a density, it is non-negative. For the proof of the boundedness from above and we follow for example [10, Relation (*) p. 176]. Fix ω in Ω . Let $E := \{s \in [0, T], 1 - \varphi_i(s)(\omega) < 0\}$. On the one hand, since $d(\mu^1 + \mu^2)_s$ is a positive measure, it is clear that $\int_E (1 - \varphi_i(s)) d(\mu_s^1 + \mu_s^2) \leq 0$. On the other hand we have that

$$\begin{aligned} \int_E (1 - \varphi_i(s)) d(\mu_s^1 + \mu_s^2) &= (\mu^1 + \mu^2)(E) - \int_E \varphi_i(s) d(\mu_s^1 + \mu_s^2) \\ &= (\mu^1 + \mu^2)(E) - \mu^1(E) = \mu^2(E) \geq 0. \end{aligned}$$

Thus $(\mu^1 + \mu^2)(E) = 0$ and so $\varphi_i \leq 1$, ν -a.e.. Recall that

$$dC_s = \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)} \rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s \right)^2}.$$

We have for $t \in [0, T]$

$$\begin{aligned} & \int_t^T f(s, Y_s, Z_s q_s^*) dC_s + \frac{\kappa}{2} \int_t^T d\langle L, L \rangle_s \\ &= \int_t^T f(s, Y_s, Z_s q_s^*) dC_s + \frac{\kappa}{2} \int_t^T U_s^2 d\langle N, N \rangle_s \\ &= \int_t^T \tilde{f}(s, Y_s, Z_s (\tilde{q}_s^*)_{1,1}) \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)} \rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2} \\ & \quad + \int_t^T g(s, U_s) \frac{d\langle N, N \rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2}, \end{aligned}$$

where for $s \in [t, T]$

$$\tilde{f}(s, y, z) := \begin{cases} f(s, y, z \varphi_1(s)^{-1/2}) \times \frac{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s \right)^2}, & \text{if } \varphi_1(s) \neq 0, \\ f(s, y, 0) \times \frac{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s \right)^2}, & \text{if } \varphi_1(s) = 0 \end{cases}$$

and

$$g(s, u) := \frac{\kappa}{2} u^2 \left(1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2 \right).$$

With this definition we have that $f(s, Y_s, Z_s q_s^*) = \tilde{f}(s, Y_s, (\tilde{Z}_s \tilde{q}_s^*)_1)$. Hence

$$\begin{aligned} & \int_t^T f(s, Y_s, Z_s q_s^*) dC_s + \frac{\kappa}{2} \int_t^T d\langle L, L \rangle_s \\ &= \int_t^T \left(\tilde{f}(s, Y_s, Z_s (\tilde{q}_s^*)_{1,1}) \varphi_1(s) + g(s, U_s) \varphi_2(s) \right) \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)} \rangle_s + d\langle N, N \rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right)^2} \\ &= \int_t^T \left(\tilde{f}(s, Y_s, Z_s (\tilde{q}_s^*)_{1,1}) \varphi_1(s) + g(s, U_s) \varphi_2(s) \right) d\tilde{C}_s. \end{aligned}$$

As a consequence, letting

$$\begin{aligned} h(s, Y_s, \tilde{Z}_s \tilde{q}_s^*) &:= \tilde{f}(s, Y_s, (\tilde{Z}_s \tilde{q}_s^*)_1) \varphi_1(s) + g(s, (\tilde{Z}_s \tilde{q}_s^*)_2) \\ &= \tilde{f}(s, Y_s, \tilde{Z}_s (\tilde{q}_s^*)_{1,1}) \varphi_1(s) + g(s, (\tilde{Z}_s \tilde{q}_s^*)_2) \end{aligned}$$

we obtain that (A.1) can be written as

$$Y_t = B - \int_t^T Z_s dM_s - \int_t^T U_s dN_s + \int_t^T h(s, Y_s, \tilde{Z}_s \tilde{q}_s^*) d\tilde{C}_s$$

and if the initial generator f satisfies the hypothesis (H3), so does the generator h since $\varphi_1, \varphi_2, \langle M^{(i)}, M^{(j)} \rangle_T$ and $\langle N, N \rangle_T$ are bounded processes for all i, j in $\{1, \dots, d\}$. In particular h preserves the growth in the variables y, z, u . Hence we derive at BSDE (A.2).

A.2 A priori estimates

Now we assume that M itself satisfies the martingale representation theorem and we consider the following BSDE

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T f(s, Y_s, Z_s q_s^*) dC_s, \quad (\text{A.3})$$

where M, q, C are defined as in Section 2. Suppose that the terminal condition B is a bounded real-valued random variable, the generator f satisfies assumption (H4) and that (Y, Z) is a solution to (A.3). The following a priori inequality is crucial for our differentiability and representation results.

Lemma A.1. *We assume that for every $\beta \geq 1$ we have $\int_0^T |f(s, 0, 0)| dC_s \in L^\beta(\mathbb{P})$. Let $p > 1$, then there exist constants $q \in (1, \infty)$, $c > 0$ depending only on T, p and on the BMO-norm of $K \cdot M$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^{2p} \right] + \mathbb{E} \left[\left(\int_0^T |q_s Z_s^*|^2 dC_s \right)^p \right]$$

$$\leq c\mathbb{E} \left[|B|^{2pq} + \left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2pq} \right]^{\frac{1}{q}}.$$

Proof. In the following, positive constants appearing in inequalities will be denoted by c and c_1 and may change from line to line. We follow [5, Lemmata 7-8 and Corollary 9] (see also [3, Lemma 6.1]) which have been designed for the Brownian setting. However, as we will show below their arguments can be extended to the case of continuous local martingales. We proceed in several steps.

In a first step we exploit properties of BMO-martingales. Let

$$J_s = \begin{cases} \frac{f(s, Y_s, Z_s q_s^*) - f(s, 0, Z_s q_s^*)}{Y_s}, & \text{if } Y_s \neq 0, \\ 0, & \text{otherwise, and} \end{cases}$$

$$H_s = \begin{cases} \frac{f(s, 0, Z_s q_s^*) - f(s, 0, 0)}{|q_s Z_s^*|^2} Z_s, & \text{if } |q_s Z_s^*|^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then BSDE (A.3) has the form

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T (J_s Y_s + (q_s H_s^*)(q_s Z_s^*)^* + f(s, 0, 0)) dC_s, \quad t \in [0, T]. \quad (\text{A.4})$$

Due to (H4) we have $|qH^*| \leq |qK^*|$ and it follows that $H \cdot M$ is a BMO(\mathbb{P}) martingale. Furthermore we know from [15, Theorem 3.1] that there exists a $\hat{q} > 1$ such that the reverse Hölder inequality holds, i.e. there exists a constant $c > 0$ such that

$$\mathcal{E}(H \cdot M)_t^{-\hat{q}} \mathbb{E} \left[\mathcal{E}(H \cdot M)_T^{\hat{q}} | \mathcal{F}_t \right] \leq c. \quad (\text{A.5})$$

By [15, Theorem 2.3] the measure \mathbb{Q} defined by $d\mathbb{Q} = \mathcal{E}(H \cdot M)_T d\mathbb{P}$ is a probability measure. Girsanov's theorem implies that

$$\Lambda = Z \cdot M - \int_0^\cdot (q_s H_s^*)(q_s Z_s^*)^* dC_s$$

is a local \mathbb{Q} -martingale. This means that there exists an increasing sequence of stopping times $(\tau^n)_{n \in \mathbb{N}}$ converging to T such that $\Lambda_{\cdot \wedge \tau^n}$ is a \mathbb{Q} -martingale for any $n \in \mathbb{N}$. Letting $e_t = \exp(2 \int_0^t |q_s K_s^*|^{2\alpha} dC_s)$, $t \in [0, T]$, with Itô's formula applied to $e_t Y_t^2$ we have

$$\begin{aligned} d(e_t Y_t^2) &= 2|q_t K_t^*|^{2\alpha} e_t Y_t^2 dC_t + 2e_t Y_t dY_t + e_t |q_t Z_t^*|^2 dC_t \\ &= 2|q_t K_t^*|^{2\alpha} e_t Y_t^2 dC_t + 2e_t Y_t d\Lambda_t - 2e_t Y_t^2 J_t dC_t - 2e_t Y_t f(t, 0, 0) dC_t + e_t |q_t Z_t^*|^2 dC_t, \end{aligned}$$

where we used (A.4). With the inequality $J_t \leq |q_t K_t^*|^{2\alpha}$, $t \in [0, T]$, which follows from assumption (H4) we know for $t \in [0, \tau^n]$

$$e_t Y_t^2 \leq e_{\tau^n} Y_{\tau^n}^2 - \int_t^{\tau^n} 2e_t Y_t d\Lambda_t + \int_t^{\tau^n} 2e_t Y_t f(t, 0, 0) dC_t - \int_t^{\tau^n} e_t |q_t Z_t^*|^2 dC_t.$$

Note that $e_t \geq 1$ for all $t \in [0, T]$ and hence

$$e_t Y_t^2 + \int_t^{\tau^n} |q_s Z_s^*|^2 dC_s \leq e_{\tau^n} Y_{\tau^n}^2 - \int_t^{\tau^n} 2e_s Y_s d\Lambda_s + \int_t^{\tau^n} 2e_s Y_s f(s, 0, 0) dC_s. \quad (\text{A.6})$$

In a second step we provide an estimate for Y . We want to take the conditional expectation under the new measure \mathbb{Q} of the previous equality. Therefore we need to check the integrability of the involved terms. Observe that

$$e_t \leq \exp \left(2 \int_0^T |q_s K_s^*|^{2\alpha} dC_s \right), \quad t \in [0, T]. \quad (\text{A.7})$$

Using successively the monotone convergence theorem and Jensen's inequality we derive for $p > 1$

$$\mathbb{E} \left[\exp(p \int_0^T |q_s K_s^*|^{2\alpha} dC_s) \right] \leq C_T \sum_{n \geq 0} \frac{p^n}{n!} \mathbb{E} \left[\left(\int_0^T |q_s K_s^*|^2 dC_s \right)^{n\alpha} \right].$$

The Hölder inequality again along with inequality [15, p. 26] gives

$$\begin{aligned} \mathbb{E} \left[\exp(p \int_0^T |q_s K_s^*|^{2\alpha} dC_s) \right] &\leq c \sum_{n \geq 0} \frac{p^n}{n!} \mathbb{E} \left[\left(\int_0^T |q_s K_s^*|^2 dC_s \right)^n \right]^\alpha \\ &\leq c \sum_{n \geq 0} \frac{(p \|K \cdot M\|_{BMO_2}^{2\alpha})^n}{n!^{1-\alpha}} < \infty. \end{aligned} \quad (\text{A.8})$$

Thus the process e belongs to $\mathcal{S}^p(\mathbb{P})$ for all $p \geq 1$ and using the Hölder inequality and formula (A.5) we see that $e_{\tau_n} Y_{\tau_n}^2$, $e_T |B|^2$ and $\int_0^T 2e_t |Y_t| |f(t, 0, 0)| dC_t$ is in $\mathbb{L}^p(\mathbb{Q})$ for all $p \geq 1$. In the same way we get the integrability of $\int_0^{\tau_n} 2e_s |Y_s| d\Lambda_s$. Hence we are allowed to take the conditional expectation in (A.6) on both sides.

$$e_t Y_t^2 \leq \mathbb{E}^{\mathbb{Q}} \left[e_{\tau_n} Y_{\tau_n}^2 + \int_0^T 2e_s |Y_s| |f(s, 0, 0)| dC_s | \mathcal{F}_t \right].$$

Now we let n tend to infinity

$$\begin{aligned} e_t Y_t^2 &\leq \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \left[e_{\tau_n} Y_{\tau_n}^2 + \int_0^T 2e_s |Y_s| |f(s, 0, 0)| dC_s | \mathcal{F}_t \right] \\ &\leq \mathbb{E}^{\mathbb{Q}} \left[e_T |B|^2 + \int_0^T 2e_s |Y_s| |f(s, 0, 0)| dC_s | \mathcal{F}_t \right], \end{aligned}$$

where we may apply the dominated convergence theorem because of (A.7). The Young inequality with a constant $c_1 > 0$ gives

$$\begin{aligned} Y_t^2 &\leq \mathbb{E}^{\mathbb{Q}} \left[\frac{e_T}{e_t} |B|^2 + 2 \int_0^T \frac{e_s}{e_t} |Y_s| |f(s, 0, 0)| dC_s | \mathcal{F}_t \right] \\ &\leq \mathbb{E}^{\mathbb{Q}} \left[e_T |B|^2 + \frac{1}{c_1} \sup_{t \in [0, T]} |Y_t|^2 + c_1 e_T^2 \left(\int_0^T |f(s, 0, 0)| dC_s \right)^2 | \mathcal{F}_t \right] \\ &\leq \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{c_1} \sup_{t \in [0, T]} |Y_t|^2 + e_T^2 \Theta_T | \mathcal{F}_t \right], \end{aligned}$$

where we set $\Theta_T = |B|^2 + 2c_1(\int_0^T |f(s, 0, 0)|dC_s)^2$ and we take into account that $e_s/e_t \leq e_T$ for all $s, t \in [0, T]$ and $e_T \leq e_T^2$. Let $p > 1$, then we have

$$\sup_{t \in [0, T]} |Y_t|^{2p} \leq \sup_{t \in [0, T]} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{c_1} \sup_{t \in [0, T]} |Y_t|^2 + e_T^2 \Theta_T | \mathcal{F}_t \right]^p.$$

We apply Doob's inequality to obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} |Y_t|^{2p} \right] &\leq c \mathbb{E}^{\mathbb{Q}} \left[\left(\mathbb{E} \left[\frac{1}{c_1} \sup_{t \in [0, T]} |Y_t|^2 + e_T^2 \Theta_T | \mathcal{F}_T \right] \right)^p \right] \\ &\leq c \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{c_1^p} \sup_{t \in [0, T]} |Y_t|^{2p} + e_T^{2p} \Theta_T^p \right], \end{aligned}$$

and choosing c_1 such that $c/c_1^p < 1$ we have

$$\mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} |Y_t|^{2p} \right] \leq c \mathbb{E}^{\mathbb{Q}} \left[e_T^{2p} \Theta_T^p \right]. \quad (\text{A.9})$$

In Step 3 we give an estimate on Z under the measure \mathbb{Q} . For $p > 1$ we deduce from (A.6)

$$\left(\int_0^{\tau^n} |q_s Z_s^*|^2 dC_s \right)^p \leq c \left(|e_{\tau^n} Y_{\tau^n}^2|^p + \left| \int_0^{\tau^n} e_s Y_s d\Lambda_s \right|^p + \left(\int_0^T 2e_s |Y_s| |f(s, 0, 0)| dC_s \right)^p \right).$$

Then the Burkholder-Davis-Gundy and two times Young inequality (with constants $\tilde{c}_1, \tilde{c}_2 > 0$) imply

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^{\tau^n} |q_s Z_s^*|^2 dC_s \right)^p \right] \\ &\leq c \left(\mathbb{E}^{\mathbb{Q}} \left[e_T^p \sup_{t \in [0, T]} |Y_t|^{2p} \right] + \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^{\tau^n} e_s^2 Y_s^2 |q_s Z_s^*|^2 dC_s \right)^{\frac{p}{2}} \right] \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T 2e_s |Y_s| |f(s, 0, 0)| dC_s \right)^p \right] \right) \\ &\leq c \left(\mathbb{E}^{\mathbb{Q}} \left[e_T^p \sup_{t \in [0, T]} |Y_t|^{2p} \right] + \mathbb{E}^{\mathbb{Q}} \left[(\tilde{c}_1 + \tilde{c}_2) e_T^{2p} \sup_{t \in [0, T]} |Y_t|^{2p} \right] + \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{\tilde{c}_1} \left(\int_0^{\tau^n} |q_s Z_s^*|^2 dC_s \right)^p \right] \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{\tilde{c}_2} \left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2p} \right] \right), \end{aligned}$$

and because $e_T^p \leq e_T^{2p}$ and Fatou's Lemma we have

$$\mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |q_s Z_s^*|^2 dC_s \right)^p \right] \leq c \left(\mathbb{E}^{\mathbb{Q}} \left[e_T^{2p} \sup_{t \in [0, T]} |Y_t|^{2p} \right] + \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2p} \right] \right).$$

We use the Hölder inequality with $r \geq 1$, the estimate (A.9) and the Hölder inequality with $k \geq 1$ again to deduce

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |q_s Z_s^*|^2 dC_s \right)^p \right] \\
& \leq c \left(\mathbb{E}^{\mathbb{Q}} \left[e_T^{\frac{2pr}{r-1}} \right]^{\frac{r-1}{r}} \mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} |Y_t|^{2pr} \right]^{\frac{1}{r}} + \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2p} \right] \right) \\
& \leq c \left(\mathbb{E}^{\mathbb{Q}} \left[e_T^{\frac{2prk}{r-1}} \right]^{\frac{k-1}{rk}} \mathbb{E}^{\mathbb{Q}} \left[\Theta^{prk} \right]^{\frac{1}{rk}} + \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2p} \right] \right) \\
& \leq c \mathbb{E}^{\mathbb{Q}} \left[|B|^{2prk} + \left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2prk} \right]^{\frac{1}{rk}}. \tag{A.10}
\end{aligned}$$

Here we applied (A.8) and in the last inequality we employ the Hölder inequality with exponent rk to the second summand in order to obtain the last estimate. We utilize the Hölder inequality with rk to (A.9) and hence have

$$\mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} |Y_t|^{2p} \right] \leq c \mathbb{E}^{\mathbb{Q}} \left[|B|^{2prk} + \left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2prk} \right]^{\frac{1}{rk}}. \tag{A.11}$$

In step 4 we finally want to take the expectation under the measure \mathbb{P} . Let us define $\hat{M}_t = M_t - \int_0^t H_s d\langle M, M \rangle_s$ and note that since $H \cdot M$ is a $\text{BMO}(\mathbb{P})$ martingale the process $H \cdot \hat{M}$ and hence $-H \cdot \hat{M}$ are $\text{BMO}(\mathbb{Q})$ martingales (see [15, Theorem 3.3]). Furthermore by [15, Theorem 3.1] there exist a $w, w' > 1$ such that $\mathcal{E}(H \cdot M)_T \in L^w(\mathbb{P})$ and $\mathcal{E}(-H \cdot \hat{M})_T \in L^{w'}(\mathbb{Q})$. As $\mathcal{E}(H \cdot M)^{-1} = \mathcal{E}(-H \cdot \hat{M})$ we have $d\mathbb{P} = \mathcal{E}(-H \cdot \hat{M})_T d\mathbb{Q}$. Now using the Hölder inequality with the conjugate exponent v of w (and v' of w') and estimate (A.11) we deduce

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^{2p} \right] &= \mathbb{E}^{\mathbb{Q}} \left[\mathcal{E}(-H \cdot \hat{M})_T \sup_{t \in [0, T]} |Y_t|^{2p} \right] \\
&\leq \mathbb{E}^{\mathbb{Q}} \left[\mathcal{E}(-H \cdot \hat{M})_T^{w'} \right]^{\frac{1}{w'}} \mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} |Y_t|^{2pv'} \right]^{\frac{1}{v'}} \\
&\leq c \left(\mathbb{E}^{\mathbb{Q}} \left[|B|^{2pv'rk} + \left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2pv'rk} \right]^{\frac{1}{rk}} \right)^{\frac{1}{v'}} \\
&\leq c \mathbb{E} [\mathcal{E}(H \cdot M)^w]^{\frac{1}{w}} \mathbb{E} \left[|B|^{2pvv'rk} + \left(\int_0^T |f(s, 0, 0)| dC_s \right)^{2pvv'rk} \right]^{\frac{1}{rkvv'}}.
\end{aligned}$$

Setting $q = vv'rk$ and treating estimate (A.10) similarly gives the desired result. \square

A.3 Additional material on Markov processes

We now provide an example where the function u in Theorem 3.4 does not depend trivially on M .

Let $M := (M_t)_{t \in [0, T]}$ be a continuous martingale with non-independent increments that is also a Markov process with respect to a filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Let $X := (X_t)_{t \in [0, T]}$ be the solution of the SDE

$$dX_t = \int_0^t \sigma(a, X_a) dM_a, \quad t \in [0, T] \text{ and } X_0 = 0,$$

with

$$\sigma(a, x) = \begin{cases} 1 + x, & \text{if } a \geq \frac{T}{2} \\ 0, & \text{if } a < \frac{T}{2}. \end{cases}$$

Note that the coefficient σ is Lipschitz in x for every a and that it is right continuous with left limits in a for every x ; as a consequence X admits an unique solution by [22, V. Theorem 35]. Consider a simple BSDE of the form (3.7) with $f \equiv 0$, $\kappa = 0$ and $F(x) := \log(1 + x)$. Note that F is not bounded but in this special case the existence of a solution to the BSDE may be constructed directly. Our aim is to show that $\mathbb{E}[F(X_T)|\mathcal{F}_t]$ is not a trivial function of M for $t \in [0, T]$. By Itô's formula we have

$$F(X_T) = \log(1 + X_t) + \int_t^T (1 + X_s)^{-1} dX_s - \frac{1}{2}(\langle M, M \rangle_T - \langle M, M \rangle_{t \vee \frac{T}{2}}),$$

and hence

$$\begin{aligned} \mathbb{E}[F(X_T)|\mathcal{F}_t] &= \log(1 + X_t) - \frac{1}{2}\mathbb{E}[\langle M, M \rangle_T - \langle M, M \rangle_{t \vee \frac{T}{2}}|\mathcal{F}_t] \\ &= \log(1 + X_t) - \frac{1}{2}\mathbb{E}[M_T^2 - M_{t \vee \frac{T}{2}}^2|\mathcal{F}_t] \\ &= \log(1 + X_t) - \frac{1}{2}\mathbb{E}[M_T^2 - M_{t \vee \frac{T}{2}}^2|M_t] \end{aligned}$$

since M is a Markov process. Choose $0 < t < \frac{T}{2}$ and then by definition of X , the last term on the right hand side above cannot be expressed as a trivial deterministic function of (t, X_t) since M_s cannot be deduced from X_s for $s < \frac{T}{2}$. However this term is deterministic and only depends on t if M has independent increments. This gives an example of a situation where the function u in Theorem 3.4 does not depend trivially on M .

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